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CONNES–KREIMER–EPSTEIN–GLASER RENORMALIZATION

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Abstract

Causal perturbative renormalization within the recursive Epstein–Glaser scheme involves extending, at each order, time-ordered operator-valued distributions to coinciding points. This is achieved by a generalized Taylor subtraction on test functions, which is transposed to distributions. We show how the Epstein–Glaser recursive construction can, by means of a slight modification of the Hopf algebra of Feynman graphs, be recast in terms of the new Connes–Kreimer algebraic setup for renormalization. This is illustrated for ϕ^4_4-theory.

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1 Introduction

A quarter of a century ago, the venerable art of bypassing infinities seemed poised to reach the status of science. The "state of the art" at that time was summarized in the authoritative lectures at the 1975 Erice Majorana School, including outstanding treatments of dimensional renormalization, BPHZ renormalization, the forest formula, the Epstein–Glaser scheme and the BRS method for gauge models. That School's book [1] remains an indispensable classical in the field. It is fair to say that, despite some technical and conceptual advances, no major progress in the systematics of renormalization theories took place again until recent dates.

Ever since Kreimer perceived the existence of a Hopf algebra lurking behind the forest formula [2], however, the question of encoding the systematics of renormalization in such a structure, and of the practical advantages therein, brought the subject to the forefront again. The first attempt used an algebra of parenthesized words or rooted trees to describe Feynman diagrams, and, as quickly pointed out by Krajewski and Wulkenhaar [3], although able in principle to deal with overlapping divergences (see, for instance, the Appendix of [4]), was ill adapted in practice to do so. More recently, Connes and Kreimer [5–7] were indeed able to show, using ϕ^3_6 as an example, that dimensional regularization of quantum field theories in momentum space, and the renormalization group, are encoded in a commutative Hopf algebra *of Feynman graphs* $\mathcal{H}$ and an associated Riemann–Hilbert problem. Moreover, there exists a Hopf algebra map from the Connes–Moscovici algebra $\mathcal{H}_{\rm CM}$ [8] of the group $\mathrm{Diff}'(\mathbb{C})$ of nontrivial diffeomorphisms of the complex plane to $\mathcal{H}$.

After the achievement by Connes and Kreimer, a reformulation of renormalized, perturbative quantum field theory in Hopf algebraic terms is in the cards. Because $\mathcal{H}$ is commutative, its vast group of characters $\mathcal{G}_\mathcal{H}$ contains the same information, and there is a corresponding morphism of groups $\mathcal{G}_\mathcal{H} \to \mathrm{Diff}'(\mathbb{C})$. Moreover, the Lie algebra $\mathrm{Lie}(\mathcal{G}_\mathcal{H})$ of infinitesimal characters, with bracket operation given by diagram insertions [5], and its enveloping algebra, also contain in principle the same information. Therefore, beyond its mathematical elegance and this tantalizing connection to diffeomorphism-invariant geometry in noncommutative geometry, the Hopf algebra approach holds the promise of a fruitful use of $\mathcal{G}_\mathcal{H}$-invariance as a tool of the renormalized theory [6, 7, 9].

The present paper wants to contribute to this program by considering a different renormalization setup. Dimensional renormalization in effect occupies a privileged rank among methods that use an intermediate regularization. On the other hand, the most rigorous regularization-free method is causal perturbation theory (CPT), i.e., the scheme devised by Epstein and Glaser —see [10, 11]— following the basic ideas of Stueckelberg–Bogoliubov–Shirkov [12,13], linking the problem of ultraviolet divergences to questions in distribution theory. This is not the place to go into the respective merits of regularization-bound and regularization-free methods. Suffice it to say that CPT used to have a reputation for difficulty; but nowadays its tremendous power and flexibility are widely recognized. For instance, differential renormalization [14], especially in the elegant form given to it in [15], has been shown in the remarkable paper [16] as another instance of CPT. Also BPHZ renormalization, in spite of its very different flavour, is just a Fourier space translation of a particular case of CPT [16, 17].

Being a "real" space procedure, CPT is the natural candidate for renormalization in curved back-

grounds [18]. Also, the kinship of CPT to the Segal–Shale–Stinespring formulation of quantum field theory (i.e., the non-perturbative approach based on the metaplectic and spin representations), spelled out by Bellissard in the seventies [19], then apparently forgotten, has lately [20] been highlighted again.

The aim of this paper is to show that the Connes–Kreimer Hopf algebra approach is consistent with the Epstein–Glaser renormalization method. We also proceed mainly by way of example and, in particular, we exhibit a Hopf algebra that encodes the renormalization of ϕ^4_4 scalar theory. Now, in a recent paper by Pinter [21], in the same context of ϕ^4_4, the successive renormalization of superficial divergences and subdivergences, buried in the abstract CPT procedure, was brought to light for diagrams occurring in the expansion of the scattering matrix up to third order. The present article extends Pinter's work to combinatorially nontrivial cases, as well as pursuing the work of [5–7]. The equivalence between the classical recursive formulae for renormalization and the forest formula of Zimmermann, which is shown to hold in the Epstein–Glaser procedure, boils down to the result that the "C-map" and thus the "R-map" are in fact characters of our Hopf algebra.

We do not suppose that the reader is familiar with CPT. Accordingly, some questions of principle relative to the conceptual and mathematical status of the method are reviewed in Section 2 of the paper. Sections 3 and 4 deal with the machinery of CPT, in the variant proposed by Stora. Section 5 is concerned with the multiplicative property of the counterterm map within causal perturbation theory. In Section 6, we introduce an algebra structure $\mathcal{H}$ à la Connes–Kreimer for the ϕ^4_4 theory. The renormalization method of Bogoliubov–Epstein–Glaser is then restated in the Hopf algebra context. In Section 7, our treatment of CPT is further illustrated by a few more examples. The conclusions follow.

We do suppose the reader to be familiar with the axioms of Hopf algebra theory, as, for instance, in Chapter 3 of [22] or in Chapter 1 of [23], and with the basics of Bogoliubov's R-map [13].

2 Motivating causal perturbative renormalization

In CPT the emphasis is laid not so much on eliminating the infinities apparent in the naïve bare Feynman amplitudes as on redefining the objects of the theory in such a way that infinities are never met in the first place. Thus, there can be some contention on whether CPT is truly a renormalization theory in the ordinary sense. An example, taken originally from [16], will, we hope, illuminate the matter.

In classical electrostatics, the electric potential V and field $\vec{E}$ are given as distributions on $\mathbb{R}^3$. If there is a point charge source e at the origin, the solution of the Poisson equation $\Delta V = -4\pi e\delta$ is given by the distribution e/r, an element of $\mathcal{D}'(\mathbb{R}^3)$. The electric field is obtained thus:

$$\vec{E}(\vec{x}) = -\vec{\nabla}V = \frac{e\,\vec{x}}{r^3} \in \mathcal{D}'(\mathbb{R}^3). \tag{1}$$

Now, suppose we are asked what is the total energy stored in the field. The *density* of energy would notionally be given by $|\vec{E}|^2$. However, in general distributions cannot be squared, and in particular it is clear that this product does not exist as an element of $\mathcal{D}'(\mathbb{R}^3)$. One could take the attitude of

saying that the self-energy is infinite; or one could say, perhaps more accurately, that the self-energy functional is undefined globally from physical principles, in this field theory. On the other hand,

$$|\vec{E}|^2(\vec{x}) = \frac{e^2}{r^4} \tag{2}$$

makes sense as an element of $\mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$. If we insist on seeking a sensible answer to the question, we can try to make an *extension* of r^{-4} to a distribution defined on the whole of $\mathbb{R}^3$. This is certainly possible, albeit in a nonunique way.

A bit more generally, suppose that, as in the present case, f is defined as a regular function having only an algebraic singularity of finite order at the origin. Then, a canonical way [10] of making the extension of f to a distribution defined on all of $\mathbb{R}^3$ is obtained by choosing an auxiliary function w such that $\mathrm{w}(0) = 1$ and all its derivatives up to the order of singularity $\omega(f)$ of f vanish at the origin, and then defining the extension f_{w} by subtraction and transposition:

$$\langle f_{\mathrm{w}}(\vec{x})\,,\,\varphi(\vec{x})\rangle := \langle f(\vec{x})\,,\,\varphi_{\mathrm{w}}(\vec{x})\rangle := \left\langle f(\vec{x}), \varphi(\vec{x}) - \mathrm{w}(\vec{x}) \sum_{|a|=0}^{\omega(f)} \frac{\partial_a \varphi(0)}{a!}\, x^a \right\rangle, \tag{3}$$

where a is a multi-index. This is CPT renormalization in a nutshell.

Naturally, the extension so defined depends on w, albeit in a relatively weak way. In our case, $f = |\vec{E}|^2$ and because $\omega(f) = 1$ (i.e., it would be necessary to multiply $|\vec{E}|^2(\vec{x})$ as an integrand by $r^{1+\varepsilon}$ to get a finite integral), the nonuniqueness of the extension is expressed by the possibility of adding distributions supported at the origin, of the form $C^0\delta(\vec{x}) + C^i\partial_i\delta(\vec{x})$, to any particular extension. The constants C^0, C^i parametrize the indeterminacy in the mathematical solution to the problem.

Actually, in the present case we can readily discard the three constants C^i, for reasons of symmetry. Moreover, in view of the good "infrared behaviour" of $|\vec{E}|^2$, it is possible to choose for w the natural candidate $\mathrm{w} \equiv 1$. This gives for the self-energy of the classical electron field:

$$\langle |\vec{E}|^2_1\,,\,1\rangle + C^0 = C^0. \tag{4}$$

According to this disposition, the energy is not so much infinite as undetermined. One can then try to fix the value of C^0 by extra physical information and/or assumptions. For instance, if we have measured the mass of the point charge and we believe all this mass to be of electrical origin, we must then believe that Nature selects $C^0 = m$, in standard units.

Now, one can take the attitude that, even if no infrared problem is present, it is more natural to make w different from zero only in a neighbourhood of the origin. Take, for instance, $\mathrm{w}(\vec{x}) = \Theta(a^2 - |\vec{x}|^2)$, where Θ denotes the Heaviside step function ($\Theta(t) = 1$ for $t \geq 0$ and 0 otherwise). Then one reproduces the previous result *directly* by choosing $a = 4\pi e^2/m$.

There are several morals to the story, already: continuation of distributions corresponds to renormalization; "mass scales" do appear naturally in the process; sometimes, physical requirements allow to lessen or sometimes completely eliminate the indeterminacy of renormalization.

Now, CPT-like renormalization is not the only one available on the market. In a more technical vein, let us explore some alternatives. The extension of a function f depending only on the radial

coordinate can always be reduced to a one-dimensional problem [24]; note that in the mathematical literature, the extension process is often called "regularization", a terminology which for obvious reasons we cannot embrace here. In fact, on $\mathbb{R}^n$ with $\mathbb{S}$ the unit sphere, the equality

$$\langle f(r)\,,\,\varphi(\vec{x})\rangle_{\vec{x}} = \langle \Theta(r)r^{n-1}f(r)\,,\,\tilde{\varphi}(r)\rangle_r \tag{5}$$

holds, where

$$\tilde{\varphi}(r) := \int_{\mathbb{S}} \varphi(r\vec{u})\,d\Omega(\vec{u}) \tag{6}$$

is extended to all $r \in \mathbb{R}$ as an even function, as all its odd order derivatives vanish at the origin. Our example is thus related to possible definitions for $\Theta(x)x^{-2}$. A popular one is obtained by analytic continuation of $\Theta(x)|x|^\lambda$; in this case the pole at $\lambda = -2$ is removable. Another strategy consists in extracting the "finite part" of $\Theta(x)x^{-2}$: knowing a priori that the integral

$$G(\varepsilon) := \int_\varepsilon^\infty \frac{\varphi(x)\,dx}{x^2} \tag{7}$$

is of the form $G(\varepsilon) = G_0(\varepsilon) + b\log\varepsilon + c/\varepsilon$, where G_0 has a finite limit as $\varepsilon \downarrow 0$, the finite part is defined to be this limit. Yet another idea is to define $\Theta(x)x^{-2}$ as the distributional second derivative of $-\Theta(x)\log(x)$ —this is "differential renormalization". The diligent reader will be able to check that, for our case, the three procedures coincide. This little miracle happens when extending r^{-k} in $\mathbb{R}^n$ iff $k-n$ is odd. The coincident extensions differ from an extension in the style of (3) with $w(\vec{x}) = \Theta(a^2 - |\vec{x}|^2)$, by terms that involve the scale a and the distributions δ and δ'. An interesting question, which apparently has not received much attention, is to find a w that reproduces a given extension.

To conclude, we already mentioned that the choice $w = 1$ does nothing to improve the infrared behaviour of f. In order to obtain a good infrared behaviour, however, it is not necessary to proceed to sharp cutoffs; more precisely, it is enough that w go to zero rapidly in the Cesàro sense [25, 26]. For instance, the difference between BPHZ subtraction at zero momentum and at $q \neq 0$ momentum in CPT corresponds essentially to the replacement of 1 by an exponential function —that decreases faster than any inverse power of x in the Cesàro sense— as the w-function in real space.

The example in this section differs less in substance than in technical simplicity/complication from the treatments of the divergences that crop up in quantum field theory.

3 Surveying the Epstein–Glaser procedure

Since we are concerned with the ϕ^4_4 theory as a pedagogical example, the discussion in this section will be centred on that instance. We use the standard unit system ($\hbar = c = 1$) throughout the paper. Let M_4 be the 4-dimensional spacetime endowed with the Minkowski metric $(+,-,-,-)$. The following notation will be repeatedly used.

Notation. According to the chosen signature, for any 4-vector $\xi = (\xi^0, \vec{\xi})$ in any Lorentzian frame, with ξ^0 denoting the time component, the Minkowski metric is written $\xi^2 = (\xi^0)^2 - \vec{\xi}^2$.

Let $\overline{V}^-$ be the closure of the past light cone,

$$\xi \in \overline{V}^- \iff \xi^0 \leq 0 \text{ and } \xi^2 \geq 0 \text{ (or } \xi^0 \leq -|\vec{\xi}|). \tag{8}$$

For any pair of points $x, y \in M_4$, we shall write

$$x \leq y \iff x - y \in \overline{V}^-,$$

and for the complement,

$$x \gtrsim y \iff x - y \notin \overline{V}^- \iff x \in \complement\left(\{y\} + \overline{V}^-\right) \iff x^0 - y^0 > 0 \text{ or } (x-y)^2 < 0, \tag{9}$$

which in particular means that x belongs to an open region in M_4. Finally,

$$x \sim y \iff x \gtrsim y \;\text{ and }\; y \gtrsim x \iff (x-y)^2 < 0,$$

meaning that x and y are spacelike-separated points. It is to be also noted that

$$x \leq y \;\text{ and }\; y \leq x \implies x = y.$$

3.1 The ϕ_4^4 theory

Consider the *free* (neutral) scalar field φ of mass m over M_4 subjected to the Klein–Gordon equation

$$(\Box + m^2)\varphi = 0, \tag{10}$$

which is derivable from the action

$$S_0(\varphi) = \frac{1}{2}\int_{M_4} d^4x\, (\partial_\nu\varphi\, \partial^\nu\varphi - m^2\varphi^2)(x) =: \int_{M_4} d^4x\, \mathcal{L}_0(x). \tag{11}$$

To this classical field φ one may associate the quantum field ϕ, an operator valued distribution which acts on the bosonic Fock space $\mathcal{F}$ built on the (spin 0, mass m)-representation space of the Poincaré group, under which the classical action (11) is invariant. One has the commutation relations

$$[\phi(x), \phi(y)] = i\, \Delta_{\rm JP}(x-y) = i\Delta^+(x-y) - i\Delta^+(y-x), \tag{12}$$

where the decisive locality (or microcausality) property

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad x \sim y \tag{13}$$

stems from the causal support property of the Jordan–Pauli commutation ordinary distribution, $\operatorname{supp}\Delta_{\rm JP} = \overline{V}^+ \cup \overline{V}^-$, and where $(\Box + m^2)\Delta^+(x) = 0$. The problem is to make sense of the field at the quantum level when S_0 is supplemented with an interaction term

$$S_{\rm Int}(\varphi) = -\frac{\lambda}{4!}\int_{M_4} d^4x\, \varphi^4(x) =: \int_{M_4} d^4x\, \mathcal{L}_1(x). \tag{14}$$

The first step consists in constructing a quantum version of φ^4, namely the fourth normal power $:\phi^4:$ of ϕ, which will share with the latter the causality property

$$\big[:\phi^4(x):, :\phi^4(y):\big] = 0 \quad \text{for} \quad x \sim y. \tag{15}$$

Let us recall that the necessity of defining Wick powers arises because ϕ is a distribution and thus cannot be raised to a power. The passage from ϕ^4 to $:\phi^4:$ is achieved by a suitable renormalization

$$:\phi^4(x): \;=\; \lim_{x_1,\dots,x_4 \to x} :\phi(x_1)\cdots\phi(x_4): \tag{16}$$

where the Wick-ordered product on the right hand side is given by the usual Wick theorem for normal ordering products, e.g. [13]. More generally, one may associate to any local monomial $\mathcal{M}(\varphi,\partial\varphi)(x)$ of the classical field φ and its derivatives its Wick-ordered quantum version $:\mathcal{M}(\phi,\partial\phi):(x)$. The subtraction rule is then more complicated, but still relies on the construction of Wick-ordered products [27].

3.2 The general Epstein–Glaser construction

As is well known, the definition of the interaction dynamics for the quantum field leads to more severe renormalization problems. Heuristically, one takes the Dyson formula for the scattering matrix,

$$\mathbf{S} = T \exp\left(i \int_{M_4} d^4x\, \mathcal{L}_1(x) \right), \tag{17}$$

where T denotes the chronological product or T-product. Due to the distributional character of the free field ϕ, equation (17) is ill-defined. Our account of the idea of Stueckelberg–Bogoliubov–Shirkov [12, 13], further elaborated in [10], to overcome this problem goes as follows.

First, for $\mathcal{L}_1(x) = -\lambda\, \varphi^p(x)/p!$, consider the extended interacting Lagrangian

$$\underline{g}(x)\underline{\mathcal{L}}(x) = \sum_{j=1}^{p} g_j(x)\, \mathcal{L}^{(j-1)}(x), \tag{18}$$

where $\underline{\mathcal{L}}$ denotes collectively a finite sequence of Wick monomials of the free field and its derivatives, stable in the sense that it contains all submonomials obtained by formal derivation of the interacting Lagrangian $\mathcal{L}_1$ with respect to ϕ, so that in particular the field $\mathcal{L}^{(p-1)} = \phi$ itself is there. The g_j are Schwartz (smooth, rapidly decreasing) functions on M_4 that serve as sources to generate fields and composite fields, and also play the role of Lorentz-covariant "coupling constants". They are separated into g_1, which will be subject to the adiabatic limit, namely $g_1 \to \lambda$, and the others, which will be sent to zero (note that g_p is the classical source J of ϕ). In the instance of ϕ^4_4 theory, $\underline{\mathcal{L}}$ is given by the following sequence of Wick powers of the free field ϕ,

$$\mathcal{L}^{(k)} = -\frac{:\phi^{4-k}:}{(4-k)!} \quad \text{for} \quad k = 0, \dots, 3, \qquad \mathcal{L}^{(0)} = \frac{\mathcal{L}_1}{\lambda}, \quad \mathcal{L}^{(4)}(x) = -\mathbb{1}. \tag{19}$$

Second, one constructs a formal power series of the Gell-Mann–Low type in the g_j by defining the corresponding scattering operator $\mathbf{S}$ acting densely in $\mathcal{F}$,

$$\mathbf{S}(\underline{g}) = \mathbb{1} + i \int_{M_4} d^4x\, \underline{g}(x)\underline{\mathcal{L}}(x) + \sum_{n\geq 2} \frac{i^n}{n!} \int_{M_4^n} d^4x_1 \dots d^4x_n\; T_n(x_1, \dots, x_n)\, \underline{g}(x_1)\cdots\underline{g}(x_n), \tag{20}$$

where the T_n are operator-valued covariant symmetric distributions,

$$T_n(x_1, \dots, x_n) := T(\underline{\mathcal{L}}(x_1) \cdots \underline{\mathcal{L}}(x_n)). \tag{21}$$

These distributions will be properly defined time-ordered products of the indicated Poincaré-invariant Wick monomials, whose recursive construction is explained directly below. No convergence condition is imposed on this formal expansion. For variants of the construction, the reader is referred, as well as the original work by Epstein and Glaser [10], to the treatise [17] and to [21].

Third, following now the general strategy advocated by R. Stora [28–30], we give a concise approach to the Epstein–Glaser scheme in a geometrical setup which also applies to the Euclidean manifold situation [31].

Due to the symmetry of the distributional kernels in their arguments, we can use the following shorthand.

Notation. For n distinct points $x_i \in M_4$, $i = 1, \dots, n$, one abbreviates

$$\begin{aligned} (x_1, \dots, x_n) &= X, \qquad n = |X|, \\ \underline{g}(x_1) \cdots \underline{g}(x_n) &= \underline{g}(X), \\ T_n(x_1, \dots, x_n) &= T(X). \end{aligned} \tag{22}$$

Formal scattering theory would yield

$$T(X) = \sum_{\sigma \in \mathcal{P}_{|X|}} \left(\prod_{k=1}^{|X|-1} \Theta(x^0_{\sigma(k)} - x^0_{\sigma(k+1)}) \right) \underline{\mathcal{L}}(x_{\sigma(1)}) \cdots \underline{\mathcal{L}}(x_{\sigma(n)}), \tag{23}$$

where $\mathcal{P}_{|X|}$ denotes the group of permutations of $|X|$ objects. This naïve definition leads to the infinities of perturbation theory because the operator-valued distributions in general cannot be multiplied by the discontinuous step function. One requires instead that if the coupling constant $\underline{g}$ splits into a sum of two components with supports separated by a spacelike surface, (see Figure 1), then $\mathbf{S}(\underline{g})$ fulfils the

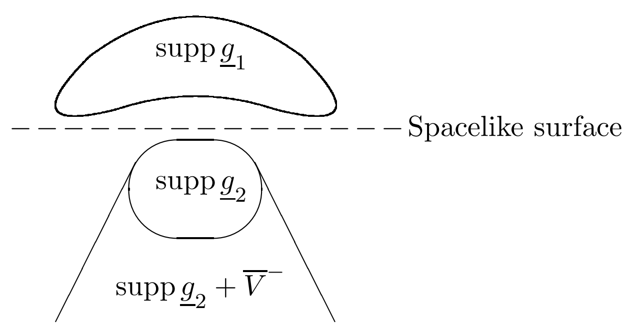


Figure 1: Causally separated domains

causal factorization property [13], which stems from locality:

$$\mathbf{S}(\underline{g}_1 + \underline{g}_2) = \mathbf{S}(\underline{g}_1)\, \mathbf{S}(\underline{g}_2) \quad \text{iff} \quad \operatorname{supp} \underline{g}_1 \gtrsim \operatorname{supp} \underline{g}_2 \iff \operatorname{supp} \underline{g}_1 \cap (\operatorname{supp} \underline{g}_2 + \overline{V}^-) = \emptyset. \tag{24}$$

This is woven into a double recursion hypothesis, as follows.

The double recursion hypothesis. Let X be a finite subset of $|X|$ distinct points of M_4 and let $M_4^{|X|}$ denote the cartesian product of $|X|$ copies of M_4. As a $|X|$-uple, X is a point in $M_4^{|X|}$. Considering a partitioning $X = I \uplus I'$ of X into two nonempty subsets I, $\emptyset \subsetneqq I \subsetneqq X$ and $I' = X \setminus I$ amounts to projecting X onto $I = (x_{i_1}, \ldots, x_{i_{|I|}})$, $(i_1, \ldots, i_{|I|}) \subsetneq (1, \ldots, |X|)$ with $1 \leq |I| \leq |X| - 1$. Let now X vary in $M_4^{|X|}$ and look to the past from the region defined by the complement I', one gets a closed wedge in $M_4^{|X|}$. Let $\mathcal{C}_I$ be the complement of this wedge, namely the generalized open cone in $M_4^{|X|}$ defined —see (9)— as

$$\mathcal{C}_I = \{\, X \in M_4^{|X|} : I \gtrsim I',\ I \uplus I' = X \,\},$$

which is translation-invariant. Then we make the following

<u>Hypothesis</u> 1 (Causal factorization) For $|X| < n$, $T(X)$ has been constructed so that

$$T(X) = T(I)\, T(I') \quad \text{in} \quad \mathcal{C}_I. \tag{25}$$

<u>Hypothesis</u> 2 (Locality) If Y is another finite subset of M_4, then for $|X| < n$ and $|Y| < n$, in the region of $M_4^{\max(|X|,|Y|)}$ defined by $X \sim Y = \{\, X \gtrsim Y \text{ and } Y \gtrsim X \,\}$,

$$[T(X), T(Y)] = 0. \tag{26}$$

One sets $T(\emptyset) = 1$. For $n = 1$, since $\underline{\mathcal{L}}$ is stable, the simplest such T_1 is $T_1 = \mathcal{L}^{(3)} = \phi$, the free field itself, which obviously satisfies the recursion hypothesis thanks to (13). It is to be remarked that although Hypothesis 1 implies $[T(I), T(I')] = 0$ in the region where $I \sim I'$, Hypothesis 2 is actually necessary in order to obtain $[\underline{\mathcal{L}}(x), \underline{\mathcal{L}}(y)] = 0$ for $x \sim y$, because at $|X| = 1$ there is no causal factorization at all. Note, moreover, that the definition of $\underline{\mathcal{L}}$ in (18) is such that all the T-products between all possible Wick monomials will be recursively constructed.

The construction of $T(X)$ for $|X| = n$ is grounded on the following lemma.

Lemma 3.1 (The geometrical lemma) *Given $|X| = n$ distinct points varying in M_4, with each partition $I \uplus I' = X$ of X into two non empty subsets, I and I', one associates in $M_4^{|X|}$ the translational invariant cone $\mathcal{C}_I = \{\, X \in M_4^{|X|} : I \gtrsim I',\ I \uplus I' = X \,\}$. This yields the open region*

$$\bigcup_{\emptyset \subsetneqq I \subsetneqq X} \mathcal{C}_I = M_4^{|X|} \setminus D_{|X|}\ , \tag{27}$$

where $D_{|X|} = \{\, x_1 = x_2 = \cdots = x_{|X|} \,\}$ is the complete diagonal in $M_4^{|X|}$.

Proof. It goes by taking the complement, namely, showing that

$$\bigcap_{\emptyset \subsetneqq I \subsetneqq X} \complement\, \mathcal{C}_I = D_{|X|}. \tag{28}$$

As a complement, $\mathcal{C}_I$ is written,

$$\mathcal{C}_I = \{\, X = I \uplus I',\ I \gtrsim I',\ I \neq \emptyset,\ I' \neq \emptyset \,\} = \bigcap_{i \in I, i' \in I'} \{x_i \gtrsim x_{i'}\} = \complement \bigcup_{i \in I, i' \in I'} \{x_i \leq x_{i'}\}. \tag{29}$$

The left hand side of (28) is written as

$$\bigcup_{\substack{i(I) \\ i'(I')}} \bigcap_{\emptyset \subsetneqq I \subsetneqq X} \{i(I) \leq i'(I')\}$$

where the union runs over all the possible assignments $I \mapsto i(I) \in I$ and $I \mapsto i'(I') \in I'$. Given such a choice, say for I', the intersection over all I is performed as follows.

- Step 1. Starting with $I = \{x_1\}$, for I' one sets $i'(I') = x_{i_2}$, this yields $x_1 \leq x_{i_2}$.
- Step 2. Taking $I = \{x_1, x_{i_2}\}$ and setting $i'(I') = x_{i_3}$ gives rise to

  $$\bigcup_{i(\{x_1, x_{i_2}\})} \Big(\{x_1 \leq x_{i_2}\} \cap \{i(\{x_1, x_{i_2}\}) \leq x_{i_3}\} \Big) = \{x_1 \leq x_{i_2}\} \cap \{x_1 \leq x_{i_3}\},$$

  whatever the choice $i(\{x_1, x_{i_2}\})$ is, thanks to step 1.
- By iterating, at the pth step, $2 \leq p \leq |X| - 1$, having obtained $\bigcap_{k=2}^{p} \{x_1 \leq x_{i_k}\}$, one takes $I = \{x_1, \dots, x_{i_p}\}$ and sets $i'(I') = x_{i_{p+1}}$, and whatever is the choice $i(\{x_1, \dots, x_{i_p}\})$ one obtains

  $$\bigcup_{i(\{x_1, \dots, x_{i_p}\})} \left(\bigcap_{k=2}^{p} \{x_1 \leq x_{i_k}\} \cap \{i(\{x_1, \dots, x_{i_p}\}) \leq x_{i_{p+1}}\} \right) = \bigcap_{k=2}^{p+1} \{x_1 \leq x_{i_k}\}.$$

  After $|X| - 1$ steps, one finally ends with $\bigcap_{k=2}^{|X|} \{x_1 \leq x_k\}$.
- One then chooses $I = \{x_2\}$, and defines $i'(I') = x_{j_2}$. Proceeding as before, one gets after at most $|X| - 1$ steps $\bigcap_{k=1, k \neq 2}^{|X|} \{x_2 \leq x_k\}$ containing the restriction $x_2 \leq x_1$. This yields

  $$\{x_1 = x_2\} \cap \bigcap_{k=3}^{|X|} \{x_1 \leq x_k\}.$$

- Going further ahead, after a finite number of steps (at most $\frac{1}{2}(|X|-1)(|X|+2)$) one has exhausted the full diagonal $D_{|X|} = \{x_1 = \cdots = x_{|X|}\}$. The latter remains stable under the intersection with all the conditions $i(I) \leq i'(I')$ coming from all unused I. $\square$

The Lemma prompts us to define on $M_4^{|X|}$ the operator-valued distributions

$$T_I(X) = T(I)\,T(I'), \qquad I \uplus I' = X, \qquad I \neq \emptyset,\ I' \neq \emptyset. \tag{30}$$

For I and J such that $\mathcal{C}_I \cap \mathcal{C}_J \neq \emptyset$, one easily proves that on $M_4^{|X|} \setminus D_{|X|}$

$$T_I(X)\Big|_{\mathcal{C}_I \cap \mathcal{C}_J} = T_J(X)\Big|_{\mathcal{C}_I \cap \mathcal{C}_J}. \tag{31}$$

Indeed, using Hypothesis 1 in both $\mathcal{C}_I$ and $\mathcal{C}_J$,

$$T(I)\,T(I') = T(I \cap J)\,T(I \cap J')\,T(I' \cap J)\,T(I' \cap J'),$$

and similarly $T(J)\,T(J') = T(J \cap I)\,T(J \cap I')\,T(J' \cap I)\,T(J' \cap I')$. The equality of both expressions follows from $[T(I \cap J'), T(I' \cap J)] = 0$ by applying Hypothesis 2 because in $\mathcal{C}_I \cap \mathcal{C}_J$, $I \cap J' \sim I' \cap J$.

Due to the stability feature of $\underline{\mathcal{L}}$ the recursion hypothesis is then reduced from operator-valued distributions to scalar distributions according to the Wick expansion formula [13] for T-products, which in our ϕ_4^4 case is typically written

$$T\left(\frac{:\phi^{p_1}(x_1):}{p_1!}\cdots\frac{:\phi^{p_n}(x_n):}{p_n!}\right) = \sum_{\substack{q_j + r_j = p_j \\ j = 1,\dots,n}} \left\langle T\left(\frac{:\phi^{r_1}(x_1):}{r_1!}\cdots\frac{:\phi^{r_n}(x_n):}{r_n!}\right)\right\rangle \frac{:\phi^{q_1}(x_1)\cdots\phi^{q_n}(x_n):}{q_1!\cdots q_n!} \tag{32}$$

for $1 \leq p_j \leq 4$; we have followed standard practice of using angle brackets to denote vacuum expectation values. In a shorthand notation, in order to be more transparent in the accounting of all the Wick monomials involved in $\underline{\mathcal{L}}$, this can be rewritten using multi-indices p, q, r with $|p| \leq 4|X|$ under the form

$$T_p(X) = \sum_{q+r=p} \langle T_r(X)\rangle \, \frac{:\phi^q(X):}{q!}\ . \tag{33}$$

We are also using "Theorem 0" of [10] which asserts that the product of a translationally invariant scalar tempered distribution given by Wick's theorem for vacuum expectation values with a Wick product is a densely defined operator on the Fock space $\mathcal{F}$. For instance, with $r_j = 1$ for all $j = 1, \dots, |X|$,

$$\langle T_r(X)\rangle = \langle T\,\mathcal{L}^{(3)}(X)\rangle = \langle T\,\phi(x_1)\cdots\phi(x_{|X|})\rangle \ = \begin{cases} 0, & |X| \text{ odd} \\ \displaystyle\sum_{\substack{\text{partitions} \\ \text{in pairs}}} \prod_{i<j} \langle T\phi(x_i)\phi(x_j)\rangle, & |X| \text{ even.} \end{cases} \tag{34}$$

Therefore (31) holds for the various scalar coefficients in (32). Then, on applying well-known results about the support of distributions, the collection of restricted operator-valued distributions

$$\overset{\circ}{T}_I(X) = T_I(X)|_{\mathcal{C}_I} \tag{35}$$

defines, thanks to Lemma 3.1, a single operator-valued distribution

$$\overset{\circ}{T}_p(X) = \sum_{q+r=p} \langle T_r(X)\rangle \, \frac{:\phi^q(X):}{q!} \tag{36}$$

in $M_4^{|X|} \setminus D_{|X|}$. For instance, for $|X| = 2$ and $|p| = 2$,

$$\overset{\circ}{T}_2(x,y) = {:}\phi(x)\phi(y){:} + \Delta(x-y), \tag{37}$$

where (23) has been used in order to define the Feynman propagator Δ,

$$\langle T\phi(x)\phi(y)\rangle = i\Theta(x^0 - y^0)\Delta^+(x-y) + i\Theta(y^0 - x^0)\Delta^+(y-x) =: \Delta(x-y), \tag{38}$$

with $i\Delta^+(x-y) = \langle\phi(x)\phi(y)\rangle$ being the contraction and $(\Box + m^2)\Delta(x) = -i\,\delta(x)$.

Renormalization then consists in finding an extension $T(X)$ of $\overset{\circ}{T}(X)$ to the whole of $M_4^{|X|}$, namely extending from the subspace of test functions whose support does not contain the diagonal $D_{|X|}$ to the full space of test functions. Suppose that such an extension $T(X)$ has been thus obtained for $|X| = n$. Then it is clear that it satisfies the double recursion hypothesis: Hypothesis 1 by construction, by virtue of (30), and Hypothesis 2 because of $T_I(X) - T_{I'}(X) = [T_I(X), T_{I'}(X)] = 0$ for $I \sim I'$ with $I \uplus I' = X$, $I \neq \emptyset$, $I' \neq \emptyset$ by taking $J = I'$ in (31). Moreover, $T(X)$ for $|X| = n$, constructed through the Wick's expansion (32), does fulfil Wick's theorem.

4 Extension of distributions in ϕ_4^4 theory

According to (33), the amplitudes $\langle T_r(X)\rangle$ are scalar tempered-distribution coefficients in the Wick expansion in terms of Wick monomials of the free field operators, belonging to $\underline{\mathcal{L}}$. For a while, the former will be denoted by $\overset{\circ}{t}_r(X)$ so that (36) is rewritten

$$\overset{\circ}{T}_p(X) \;=\; \sum_{q+r=p} \overset{\circ}{t}_r(X)\, \frac{{:}\phi^q(X){:}}{q!}\;. \tag{39}$$

By virtue of Wick's theorem and (38), $\overset{\circ}{t}_r(X)$ is a product of Feynman propagators at noncoinciding points and thus translation-invariant; it depends on $|X|-1$ independent difference variables. According to Theorem 0 of [10], extending $\overset{\circ}{T}(X)$ defined on $M_4^{|X|} \setminus D_{|X|}$ to the whole $M_4^{|X|}$ amounts to extending each of the scalar distributions $\overset{\circ}{t}(X)$ to all of $M_4^{|X|}$. As discussed in [16–18,32], an extension of a scalar distribution $\overset{\circ}{t}$ always exists, but it is of course not unique, the ambiguity being given by a distribution with support on the full diagonal $D_{|X|}$, whose degree depends on the "degree of singularity" of $\overset{\circ}{t}$. The power counting theory [10, 17, 18] then emerges in order to control these ambiguities by shifting the singularity at the origin of $\mathbb{R}^{4(|X|-1)}$, thanks to a change of variables from X to difference variables $(x_1 - x_2, \ldots, x_{|X|-1} - x_{|X|})$.

Power counting. The behaviour at the origin of $\mathbb{R}^{4n}$ of a scalar tempered distribution f can be described by both the scaling degree and the singular order; the latter is also called the power counting index.

The scaling degree σ of a scalar distribution f at the origin of $\mathbb{R}^{4n}$ is defined to be

$$f \in \mathcal{D}'(\mathbb{R}^{4n}), \qquad \sigma(f) = \inf\{\, s : \lim_{\lambda \to 0} \lambda^s f(\lambda x) = 0\,\}, \tag{40}$$

where the limit is taken in the sense of distributions. It is clearly a generalization of the notion of degree of a homogenous distribution. The concept of scaling degree is due to Steinmann [33]; its main features are summed up in the following statement of [18].

Lemma 4.1 *For $f \in \mathcal{D}'(\mathbb{R}^{4n})$ and a a multi-quadri-index,*

1. $\sigma(x^a f) = \sigma(f) - |a|$.

2. $\sigma(\partial_a f) = \sigma(f) + |a|$, *(derivation increases the scaling degree).*

3. *If* $\mathrm{w} \in \mathcal{D}(\mathbb{R}^{4n})$ *with* $\sigma(\mathrm{w}) \leq 0$*, then* $\sigma(\mathrm{w} f) \leq \sigma(f)$.

4. $\sigma(f_1 \otimes f_2) = \sigma(f_1) + \sigma(f_2)$.

The singular order ω of the scalar distribution f at the origin of R^{4n} is defined to be

$$f \in \mathcal{D}'(\mathbb{R}^{4n}), \qquad \omega(f) = [\sigma(f) - 4n], \tag{41}$$

where 4 is the spacetime dimension and $[\cdot]$ denotes integer part.

In particular, the amplitudes will be scalar tempered distributions and, for instance, the propagator $\langle T\phi(x)\phi(y)\rangle = \Delta(x-y)$ in the ϕ_4^4 theory has scaling degree $\sigma(\Delta) = +2$, and since the scaling degree of the tensor product of scalar distributions is the sum of those of the factors, in particular the scaling degree of $\prod_{k=1}^{|X|-1} \Delta(x_k - x_{k+1})$ is $2(|X|-1)$.

In $\mathbb{R}^4$, for the Dirac distribution $\sigma(\delta) = 4$, thus $\omega(\delta) = 0$, and for the following tempered distributions,

$$\omega(\Delta^+) = \omega(\Delta) = -2, \qquad \omega(\Delta(x_1 - x_2)^{|X|-1}) = 2|X| - 6. \tag{42}$$

Moreover, if $\omega(\overset{\circ}{t}) < 0$, then there is a unique scalar distribution $t \in \mathcal{D}'(\mathbb{R}^{4n})$ which extends $\overset{\circ}{t}$ from the subspace $\mathcal{D}(\mathbb{R}^{4n} \setminus \{0\})$ of test functions whose support does not contain the origin, in the sense that $\langle t\,,\,\varphi\rangle = \langle \overset{\circ}{t}\,,\,\varphi\rangle$ for any $\varphi \in \mathcal{D}(\mathbb{R}^{4n} \setminus \{0\})$.

For $\omega := \omega(\overset{\circ}{t}) \geq 0$, call $\mathcal{D}_\omega(\mathbb{R}^{4n})$ the finite-codimensional subspace of test functions vanishing up to order ω at the origin,

$$\mathcal{D}_\omega(\mathbb{R}^{4n}) := \{\, \varphi \in \mathcal{D}(\mathbb{R}^{4n}) : \partial_a \varphi(0) = 0, \ \forall\ |a| \leq \omega \,\}, \tag{43}$$

where a is a multi-quadri-index.

For all $\overset{\circ}{t} \in \mathcal{D}'_\omega(\mathbb{R}^{4n})$, there exist scalar distributions $t \in \mathcal{D}'(\mathbb{R}^{4n})$ with $\omega(t) = \omega$ and $\langle t\,,\,\varphi\rangle = \langle \overset{\circ}{t}\,,\,\varphi\rangle$ for any $\varphi \in \mathcal{D}_\omega(\mathbb{R}^{4n})$. Power counting theory asserts that there exists a *minimal class* of extensions, (in the sense that their singular orders are the smallest possible), within which the ambiguity on the extensions reduces to the form

$$\Delta t(x) = \sum_{|a|=0}^{\omega} \frac{(-1)^{|a|}\, C^a}{a!}\, \delta^{(a)}(x), \tag{44}$$

where $\langle (-1)^{|a|}\,\delta^{(a)}\,,\,\varphi\rangle = \partial_a\varphi(0)$ and where the C^a are free constants due to translation invariance, which are often fixed by symmetry (Lorentz, gauge…) considerations, with the help of Ward and/or Slavnov identities, e.g. [17,21,34,35]. Note that $\omega(\Delta t) = \omega(t)$. In particular in ϕ^4_4 theory, assume that a scalar extension has been constructed for each of the $\overset{\circ}{t}_r$ in (39) with singular degree $\omega(r) \geq 0$; then the ambiguities (44) yield an ambiguity for the extension of (39),

$$\Delta T_p(X) = \sum_{q+r=p} P_r(\partial)\,\delta(x_1 - x_2)\cdots\delta(x_{|X|-1} - x_{|X|})\,\frac{:\!\phi^q(X)\!:}{q!} \tag{45}$$

where each of the polynomials of derivatives (with constant coefficients)

$$P_r(\partial) \;=\; \sum_{|a|=0}^{\deg P_r} \frac{(-1)^{|a|}\,C^a}{a!}\;\partial_a \tag{46}$$

has (maximal) degree $\omega(r)$ given by power counting according to

$$\deg P_r \leq \omega(r) = \omega_X - \omega\Big(\frac{:\!\phi^q\!:}{q!}\Big), \qquad \omega_X = \sum_{j=i}^{|X|}\bigg(\omega\Big(\frac{:\!\phi^{p_j}\!:}{p_j!}\Big) - 4\bigg) + 4. \tag{47}$$

The power counting index (or degree) ω for Wick monomials is defined to be

$$\omega\Big(\frac{:\!\phi^q\!:}{q!}\Big) = \omega(\phi)\,|q|. \tag{48}$$

For the case of interest of a massive scalar free field, $\omega(\phi) = 1$, coming from

$$\omega(\phi) \;=\; \tfrac{1}{2}(4 - \deg K_0), \tag{49}$$

a formula which stems from Lemma 4.1 together with the requirement that the singular degree of the free Lagrangian $\mathcal{L}_0 = :\!\phi\,K_0\phi\!:$ be 4.

Notice now that $\langle T_r(X)\rangle$ is represented as a sum of Feynman graphs Γ (connected or not) with $|X|$ vertices and $|q| = |p| - |r|$ external lines, where q_j external lines are attached to the jth vertex of type $\mathcal{L}^{(4-p_j)}$, $j = 1, \ldots, |X|$, so that $\omega(r) = \sum_{j=i}^{|X|}\bigg(\omega\Big(\frac{:\!\phi^{r_j}\!:}{r_j!}\Big) - 4\bigg) + 4$ is nothing but the *superficial degree of divergence* $\omega(\Gamma)$ of the corresponding graphs Γ giving the UV behaviour, e.g. [36].

For instance, according to (33), if $|p| = 4|X| = |r| + |q|$, i.e., all the $|X|$ vertices are of the type $\mathcal{L}^{(0)} = -:\!\phi^4\!:/4!$, owing to (47) one has $\omega(\Gamma) = 4 - E$, where $E = |q|$ is the number of external lines associated to the bosonic field ϕ. So in ϕ^4_4 theory, only graphs with 2 or 4 external legs will be superficially divergent. (Vacuum graphs ($E = 0$) are dropped because they have been shown to vanish in the adiabatic limit [17,34].) Note that in their definition, the amplitudes $\langle T_r(X)\rangle$ require the knowledge of the various T-products $T_r(X)$ between all the Wick monomials belonging to $\underline{\mathcal{L}}$.

We shall now concentrate in full generality on a given graph Γ with $|X|$ vertices which contributes to the amplitude $\langle T_r(X)\rangle$ (a tempered scalar distribution) and set $f(\Gamma) := \overset{\circ}{t}_r(X)$. It remains to recall

how an extension of $f(\Gamma)$ can be exhibited by a Taylor-like subtraction, anticipated in the heuristic discussion in Section 2, the so-called W-operation [10, 17].

According to both a given graph Γ containing $n = |X|$ vertices and having power counting index $\omega := \omega(\Gamma)$, and a choice of a function w_Γ of $n-1$ variables such that $w_\Gamma(0) = 1$ and $\partial_a w_\Gamma(0) = 0$ with a a multi-quadri-index such that $1 \le |a| = |a_1| + \cdots + |a_{n-1}| \le \omega$, one defines the mapping

$$W_\Gamma \;:\; \mathcal{D}(M_4^n) \;\longrightarrow\; \mathcal{D}_\omega(M_4^n), \qquad \varphi \longmapsto W_\Gamma\, \varphi$$

where

$$(W_\Gamma\, \varphi)\,(X) = \varphi(X) - w_\Gamma(x_1 - x_2, \ldots, x_{n-1} - x_n) \sum_{|a|=0}^{\omega} \frac{(x_1 - x_2)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}}}{a!}\, \partial_a \varphi|_{D_{|X|}}. \tag{50}$$

Lemma 4.2 *For each superficially divergent graph* Γ*,* $\omega(\Gamma) \ge 0$*,* W_Γ *is a projector,* $W_\Gamma(1 - W_\Gamma) = 0$*.*

A subtraction operation can be defined by $S_\Gamma := 1 - W_\Gamma$.

In the Epstein–Glaser scheme, to any choice $I \subseteq X$ of vertices of Γ there corresponds a subgraph γ of Γ.

Lemma 4.3 *For any subgraph* $\gamma \subseteq \Gamma$ *with* $|I|$ *vertices,* $|I| \le |X|$ *and* $\omega(\gamma) \ge \omega(\Gamma)$*, one has* $S_\Gamma W_\gamma = 0$*.*

By transposition one defines the action of these projectors on tempered distributions. We keep then the same names for them; but beware that the order of operators in formulae like those of the lemma is inverted.

Of course the extension may be fixed if desired; namely, for a given set of the C constants and a given w in (44), there is a unique extension $\tilde{f}(\Gamma)$ such that

$$\begin{aligned}
&\langle \tilde{f}(\Gamma)\,,\, \varphi \rangle = \langle f(\Gamma)\,,\, W_\Gamma \varphi \rangle, \qquad \forall\, \varphi \in \mathcal{D}(M_4^n)\\
&\langle \tilde{f}(\Gamma)\,,\, (x_1 - x_2)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}}\, w_\Gamma(x_1 - x_2, \ldots, x_{n-1} - x_n) \rangle = C^a, \quad 0 \le |a| \le \omega(\Gamma).
\end{aligned}$$

Thanks to the causal factorization property involved in the definitions (30) and (35), the chronological products $T(I)\, T(I')|_{\mathcal{C}_I}$ have already been extended to subdiagonals of $M_4^{|X|}$ by construction. We call the extension to these subdiagonals the *renormalization of subdivergences* and the extension to the complete diagonal in the last step the *renormalization of the overall divergence*. The known subtraction-transposition procedures provide one way to solve this extension problem. All terms in ϕ_4^4 theory of $\mathbf{S}$ up to order 3 in the vertices can be found in [21], to which the reader is referred. We also remit to [21] for the bit of microlocal theory required to cross-check that the chronological products are well defined. It is emphasized that in the CPT scheme grounded on locality the arduous problem of overlapping divergences is disentangled in a recursive way within an operator formalism before the adiabatic limit is taken.

A criterion [10,30]. The vacuum expectation values of operator-valued distributions are conveniently represented by Feynman graphs. In order to recognize that a finite expression for a Feynman diagram

is actually a renormalization, one proceeds in configuration space by checking that in *all* the open cones $\mathcal{C}_I$ associated to the splittings $X = I \uplus I'$, $I \neq \emptyset$, $I' \neq \emptyset$, the amplitude factorizes as

$$\langle T(X)\rangle\Big|_{\mathcal{C}_I} \;=\; \langle T_I(X)\rangle\;\langle T_{I'}(X)\rangle \prod_{\substack{i\,\in\, I, i'\,\in\, I' \\ (ii')\,\in\, L}} \langle \phi(x_i)\phi(x_{i'})\rangle. \tag{51}$$

Here $\langle T_I(X)\rangle$ and $\langle T_{I'}(X)\rangle$ are renormalized amplitudes (subgraphs of respective orders $|I|$ and $|I'|$) computed at lower orders, and the product ranges over the internal lines (ii') in the set L of internal lines connecting the vertices $i \in I$ and $i' \in I'$. Formula (51) directly follows by applying Wick's theorem to the product in (36).

At this stage, an important remark (chiefly due to D. Kreimer and R. Stora) needs to be made: Hopf algebras are naturally related to problems in combinatorics and shuffle theory. There is thus a Hopf algebra structure *already* implicit in the recursive method of Epstein and Glaser, in the geometric presentation advocated here. That algebra describes the stratification of the short-distance singularities along subdiagonals [37]. However, to keep close to the treatment in [5] —and to the standard language— we formulate our Hopf algebra in terms of Feynman diagrams. This has the drawback that the algebraic method appears, although perhaps elegantly, almost as an afterthought; but we gladly pay this price in order to have a more pictorial description. For practical purposes, then, Feynman graphs serve as mnemonic devices to keep the accounting of the amplitudes and their singularities straight (for memory: vertices correspond to coupling constants, external lines to field insertions and internal lines to vacuum expectation values). Also in this connection, we should mention that there is also a Hopf algebra structure [38] behind Wick ordering, on which we lightly treaded in Section 3. We intend to return to these matters in a sequel article.

5 The counterterm map in Epstein–Glaser renormalization

We conveniently begin by collecting all the "graphological" definitions needed in this section and the next. As already mentioned, a *graph* or diagram of the theory is specified by a set of *vertices* and a set of *lines* among them; *external* lines are attached to only one vertex each, *internal* lines to two. We mentioned, once more, that diagrams with no external lines need not be taken into account. In ϕ^4_4 theory, only graphs with an even number of external lines are to be found; apart from the interaction vertices with joining of four lines, we shall consider two-line self-energy vertices corresponding to mass insertions.

Figure 2: $:\!\phi(x)^4\!:$ interaction vertex and a two-line self-energy vertex.

A diagram is connected when any two of its vertices are joined by lines, of course.

Given a graph Γ, a *subdiagram* γ of Γ is specified by a subset of at least two vertices of Γ and a subset of the lines that join these vertices in Γ. Clearly, the external lines for γ include not only the subset of original incident lines but some internal lines of Γ not included in γ. By exception, the empty subset $\emptyset$ will be admitted as a subdiagram of Γ. As well as Γ itself.

The connected pieces of Γ are the maximal connected subdiagrams. A diagram is *proper*, i.e., 1PI, when the number of its connected pieces would not increase on the removal of a single internal line; otherwise it is called *improper*, i.e., 1PR. An improper graph is the union of its proper components plus subdiagrams containing a single line. Note that in the theory at hand there are proper diagrams, like the "bikini" of Figure 3, which are however one-*vertex* reducible, i.e., they are made more disconnected on the removal of a vertex.

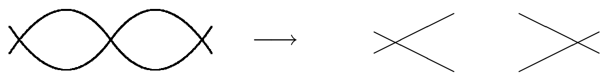


Figure 3: Example of a one-vertex reducible (though proper) graph split into a disconnected graph

A *subgraph* of a proper graph is a subdiagram that contains all the propagators that join its vertices in the whole graph; as such, it is determined solely by the vertices. A subgraph of an improper graph Γ, distinct from Γ itself, is a proper subdiagram each of whose components is a subgraph with respect to the proper components of Γ: in other words, a product of subgraphs. We write $\gamma \subseteq \Gamma$ if and only if γ is a subgraph of Γ as defined: this is the really important concept for us.

Two subdiagrams γ_1, γ_2 of Γ are said to be *nonoverlapping* when $\gamma_1 \cap \gamma_2 = \emptyset$ or $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$ in an obvious sense; otherwise they are overlapping. Given $\gamma \subseteq \Gamma$, the quotient graph or cograph Γ/γ is defined by shrinking γ in Γ to a vertex, that is to say, γ (bereft of its external lines) is considered as a vertex of Γ, and all the lines in Γ not belonging to (amputated) γ belong to Γ/γ.

Bogoliubov's R-operation is defined in the context of CPT by the W-operation of the previous section. To be precise, let $f(\Gamma) \in \mathcal{D}'(\mathbb{R}^{4k} \setminus D)$, with k the number of vertices and D any "dangerous" diagonals, the scalar distribution corresponding, by the naïve rule for the time-ordered product of propagators, to a diagram Γ contributing to the scattering matrix at order k, with E bosonic field insertions associated to the external lines. We can and shall assume here that Γ is connected, since clearly

$$f(\gamma_1 \uplus \cdots \uplus \gamma_n) = f(\gamma_1) \dots f(\gamma_n). \tag{52}$$

We know that f actually depends only on $k-1$ difference variables. If Γ is superficially divergent (i.e., if $\omega_\Gamma := \omega(f(\Gamma)) \geq 0$) and has subdivergences, we let

$$R_\Gamma f(\Gamma) = W_\Gamma \, \bar{R}_\Gamma f(\Gamma), \tag{53}$$

where the operation $\bar{R}_\Gamma$ reflects the renormalization of the subdivergences present in $f(\Gamma)$. More precisely, denoting by φ_Γ the product of the Wick monomial and the coupling constants corresponding to Γ, and keeping in mind the convention that the same symbol denotes a linear map of the theory

and its transpose,

$$\langle R_\Gamma f(\Gamma)\,,\,\varphi_\Gamma\rangle = \langle \bar{R}_\Gamma f(\Gamma)\,,\,W_\Gamma\varphi_\Gamma\rangle = \langle \bar{R}_\Gamma f(\Gamma)\,,\,(1-S_\Gamma)\varphi_\Gamma\rangle, \tag{54}$$

which, with the definition of the overall "counterterm" $C(\Gamma) := -S_\Gamma\,\bar{R}_\Gamma f(\Gamma)$, is rewritten as

$$\langle R_\Gamma f(\Gamma)\,,\,\varphi_\Gamma\rangle = \langle \bar{R}_\Gamma f(\Gamma) + C(\Gamma)\,,\,\varphi_\Gamma\rangle. \tag{55}$$

Now, Bogoliubov's recursive formula for $\bar{R}_\Gamma$ is known to be

$$\bar{R}_\Gamma f(\Gamma) := f(\Gamma) - \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} \big(S_\gamma\,\bar{R}_\gamma f(\gamma)\big)\,f(\Gamma/\gamma), \tag{56}$$

where the sum is taken over all *proper, superficially divergent, not necessarily connected subgraphs*, and $f(\Gamma/\gamma)$ is just defined by the splitting $f(\gamma)f(\Gamma/\gamma) = f(\Gamma)$. Then $\bar{R}_\gamma$ and also $C(\gamma) := -S_\gamma\bar{R}_\gamma f(\gamma)$ are recursively defined. It is clear that the Epstein–Glaser method, as described in the two previous sections fits perfectly with Bogoliubov's recursion; in fact, it was tailor-made for the purpose.

A graph without subdiagrams of the kind just defined is called *primitive*. At this point, a clarification is perhaps needed: the previous definition automatically banishes subdiagrams that result in codiagrams with "loop lines", whose initial and final vertices coincide: the so-called tadpoles. This is natural in our context, since tadpoles do not appear in the (unrenormalized) **S**-matrix expansion in real space. In particular, *the setting sun diagram is primitive* here, as it does not possesses proper *subgraphs*. Had we allowed general subdiagrams in place of subgraphs, that would not be the case. However, as explained by Hepp in [39, pp. 481–483], the added terms are irrelevant. A (nontrivial) proof that the inclusion, either uniquely of subgraphs or of general subdiagrams, in the $\bar{R}$ operation leads to equivalent results is given in the context of the parametric representation in [40]. ("Tadpoles with tail" corresponding to subdiagrams with a single external —for them— incident line do not occur at all in ϕ^4_4 theory.)

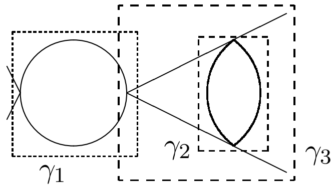
Henceforth, we shall say simply "divergent" to mean "superficially divergent".

Before tackling the main result, let us work out in full detail an example, to see how the recursive definition works in practice in the CPT framework. Suppose that the chronological product (33) has been constructed at the fourth order, and let us consider the densily defined operator $\langle T(X)\,,\,\underline{g}(X)\rangle$. Among all the possible terms in the Wick expansion we pick first the one corresponding to the graph of Figure 4. Let us choose and fix once and for all the auxiliary function w_γ for each type of superficially divergent subdiagram γ; we shall always take w even. There is a mass scale associated naturally to each of them, which we can suppose "universal" in the style of 't Hooft and Veltman, if convenient. The singular order or subtraction degree $\omega(\gamma)$ is here zero in all cases. In general, if $\gamma = \gamma_1 \uplus \cdots \uplus \gamma_n$ is a product of connected pieces, we take $w_\gamma = w_{\gamma_1}\cdots w_{\gamma_n}$. The (sub)space on which the subtraction takes place will be always explicitly displayed using coordinates. Numbering the vertices from left to right, one gets the following operator contributing to the fourth order,

$$\lambda^4\,\langle [\Delta^2(x_1-x_2)\Delta(x_2-x_3)\Delta(x_2-x_4)\Delta^2(x_3-x_4)]_R\,,\,\varphi_\Gamma(x_1,x_2,x_3,x_4)\rangle, \tag{57}$$

where $\varphi_\Gamma(x_1,x_2,x_3,x_4)$ stands for $:\!\phi^2(x_1)\phi(x_3)\phi(x_4)\!:\ g(x_1)g(x_2)g(x_3)g(x_4)$. There are four subdiagrams to consider: the fish on the left, called γ_1; the fish on the right, γ_2; the whole ice-cream cone

on the right, γ_3; and the union of the two fish, $\gamma_1 \uplus \gamma_2$. Note that γ_1 and γ_3 overlap, $\gamma_1 \cap \gamma_3 = \{x_2\}$. See again Figure 4 for these subdiagrams.



$:\phi^2(x_1)\phi(x_3)\phi(x_4):$

Figure 4: Identification of proper subdiagrams.

The cardinality of the maximal chains $\gamma_2 \subset \gamma_3 \subset \Gamma$ or $\gamma_1, \gamma_2 \subset \gamma_1 \uplus \gamma_2 \subset \Gamma$, corresponding to the maximal number of nested subdiagrams, counting Γ, is 3. This number we can call the *depth* of the diagram (the stratification of short-distance singularities is ordered by depth).

One of the things that make life with CPT truly comfortable is that, by the simple magic of duality, one actually proceeds from the top down, taking care first of the overall divergence, and then of the subdivergences as they unfold [21]. Working in coordinate space, moreover, makes rather intuitive the distribution-theoretic reasoning. There is no doubt about the soundness of the operations performed, since we work with very well-behaved Schwartz functions. With a notation that we hope is self-evident, we compute with the help of (50)

$$\begin{aligned}
&\langle [\Delta^2(x_1 - x_2)\Delta(x_2 - x_3)\Delta(x_2 - x_4)\Delta^2(x_3 - x_4)]_R\,, :\phi^2(x_1)\phi(x_3)\phi(x_4): g(x_1)g(x_2)g(x_3)g(x_4)\rangle \\
&\quad = \langle [\Delta^2(x_1 - x_2)\Delta(x_2 - x_3)\Delta(x_2 - x_4)\Delta^2(x_3 - x_4)]_{\bar R}, :\phi^2(x_1)\phi(x_3)\phi(x_4): g(x_1)g(x_2)g(x_3)g(x_4) \\
&\qquad\qquad - \, w_\Gamma(x_1 - x_2, x_2 - x_3, x_3 - x_4):\phi^4(x_4): g^4(x_4)\rangle.
\end{aligned}$$

According to (56), this is now reexpressed as

$$\begin{aligned}
&\langle \Delta^2(x_1 - x_2)\Delta(x_2 - x_3)\Delta(x_2 - x_4)\Delta^2(x_3 - x_4), :\phi^2(x_1)\phi(x_3)\phi(x_4): g(x_1)g(x_2)g(x_3)g(x_4) \\
&\quad - w_\Gamma(x_1 - x_2, x_2 - x_3, x_3 - x_4):\phi^4(x_4): g^4(x_4) - w_{\gamma_1}(x_1 - x_2):\phi^2(x_2)\phi(x_3)\phi(x_4): g^2(x_2)g(x_3)g(x_4) \\
&\qquad + w_{\gamma_1}(x_1 - x_2)\, w_\Gamma(0, x_2 - x_3, x_3 - x_4):\phi^4(x_4): g^4(x_4)\rangle \\
&\quad - \langle \Delta^2(x_1 - x_2)\Delta^2(x_2 - x_4)\Delta^2(x_3 - x_4), w_{\gamma_2}(x_3 - x_4):\phi^2(x_1)\phi^2(x_4): g(x_1)g(x_2)g^2(x_4) \\
&\qquad - w_{\gamma_2}(x_3 - x_4)\, w_\Gamma(x_1 - x_2, x_2 - x_3, 0):\phi^4(x_4): g^4(x_4)\rangle \\
&\quad - \langle \Delta^2(x_1 - x_2)[\Delta(x_2 - x_3)\Delta(x_2 - x_4)\Delta^2(x_3 - x_4)]_{\bar R}, \\
&\qquad w_{\gamma_3}(x_2 - x_3, x_3 - x_4):\phi^2(x_1)\phi^2(x_4): g(x_1)g^3(x_4) \\
&\qquad - w_{\gamma_3}(x_2 - x_3, x_3 - x_4)\, w_\Gamma(x_1 - x_2, 0, 0):\phi^4(x_4): g^4(x_4)\rangle \\
&\quad - \langle [\Delta^2(x_1 - x_2)\Delta^2(x_3 - x_4)]_{\bar R}\Delta^2(x_2 - x_4), \\
&\qquad w_{\gamma_1 \uplus \gamma_2}(x_1 - x_2, x_3 - x_4):\phi^2(x_2)\phi^2(x_4): g^2(x_2)g^2(x_4) \qquad (58) \\
&\qquad - w_{\gamma_1 \uplus \gamma_2}(x_1 - x_2, x_3 - x_4)\, w_\Gamma(0, x_2 - x_3, 0):\phi^4(x_4): g^4(x_4)\rangle. \qquad (59)
\end{aligned}$$

That corresponds to rewriting

$$\langle \bar R_\Gamma f(\Gamma)\,, W_\Gamma \varphi_\Gamma\rangle = \Big\langle f(\Gamma) - \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} \big(S_\gamma\, \bar R_\gamma f(\gamma)\big)\, f(\Gamma/\gamma), (1 - S_\Gamma)\varphi_\Gamma \Big\rangle \qquad (60)$$

as

$$\sum_{\emptyset\subseteqq\gamma\subsetneqq\Gamma}\langle\bar R_\gamma f(\gamma)\,,\,-S_\gamma f(\Gamma/\gamma)(1-S_\Gamma)\varphi_\Gamma\rangle, \tag{61}$$

where we agree that $-S_\emptyset=1$. Note that $\bar R_{\gamma_1}f(\gamma_1)=f(\gamma_1)$, $\bar R_{\gamma_2}f(\gamma_2)=f(\gamma_2)$, because these subgraphs have no subdivergences. At this stage, however, we are not yet finished, since there are still the subgraphs $\gamma_2\subset\gamma_3$ and $\gamma_1,\gamma_2\subset\gamma_1\uplus\gamma_2$ to be taken into account. When this is done, owing to (56) we obtain, in addition to the ten terms already displayed (with the last square brackets on the left hand side definitively gone), six more terms of the form

$$\begin{aligned}
&\langle\Delta^2(x_1-x_2)\Delta^2(x_2-x_4)\Delta^2(x_3-x_4),\,\mathrm{w}_{\gamma_2}(x_3-x_4)\mathrm{w}_{\gamma_3}(x_2-x_3,0){:}\phi^2(x_1)\phi^2(x_4){:}g(x_1)g^3(x_4)\\
&\quad-\quad \mathrm{w}_{\gamma_2}(x_3-x_4)\mathrm{w}_{\gamma_3}(x_2-x_3,0)\mathrm{w}_\Gamma(x_1-x_2,0,0){:}\phi^4(x_4){:}g^4(x_4)\\
&\quad+\quad \mathrm{w}_{\gamma_1}(x_1-x_2)\mathrm{w}_{\gamma_1\uplus\gamma_2}(0,x_3-x_4){:}\phi^2(x_2)\phi^2(x_4){:}g^2(x_2)g^2(x_4) && (62)\\
&\quad-\quad \mathrm{w}_{\gamma_1}(x_1-x_2)\mathrm{w}_{\gamma_1\uplus\gamma_2}(0,x_3-x_4)\mathrm{w}_\Gamma(0,x_2-x_3,0){:}\phi^4(x_4){:}g^4(x_4) && (63)\\
&\quad+\quad \mathrm{w}_{\gamma_2}(x_3-x_4)\mathrm{w}_{\gamma_1\uplus\gamma_2}(x_2-x_3,0){:}\phi^2(x_2)\phi^2(x_4){:}g^2(x_2)g^2(x_4) && (64)\\
&\quad-\quad \mathrm{w}_{\gamma_2}(x_3-x_4)\mathrm{w}_{\gamma_1\uplus\gamma_2}(x_1-x_2,0)\mathrm{w}_\Gamma(0,x_2-x_3,0){:}\phi^4(x_4){:}g^4(x_4)\rangle. && (65)
\end{aligned}$$

But note now that the terms (62) and (64) are actually equal and, in view of the assumed factorization property for $\mathrm{w}_{\gamma_1\uplus\gamma_2}$, either of them cancels the term (58) above; and analogously for terms (63) and (65) and (59), respectively. Symbolically, $(-S_{\gamma_1\uplus\gamma_2})(-S_{\gamma_1})=(-S_{\gamma_1\uplus\gamma_2})(-S_{\gamma_2})=S_{\gamma_1\uplus\gamma_2}=S_{\gamma_1}S_{\gamma_2}$, so the pattern of the subtraction, that was

$$f(\Gamma)\mapsto(1-S_\Gamma)[1-S_{\gamma_1}-S_{\gamma_2}-S_{\gamma_3}(1-S_{\gamma_2})-S_{\gamma_1\uplus\gamma_2}(1-S_{\gamma_1}-S_{\gamma_2})]\,f(\Gamma), \tag{66}$$

becomes

$$f(\Gamma)\mapsto(1-S_\Gamma)(1-S_{\gamma_1}-S_{\gamma_2}-S_{\gamma_3}(1-S_{\gamma_2})+S_{\gamma_1}S_{\gamma_2})\,f(\Gamma) \tag{67}$$

and we have to deal only with 12 terms, rather than 16.

The counterterms for the subdiagrams γ_1, γ_2, $\gamma_1\uplus\gamma_2$ can be symbolically written as

$$\begin{aligned}
C(\gamma_1)&=-\langle\Delta^2(y-x_2)\,,\,\mathrm{w}_{\gamma_1}(y-x_2)\rangle_y\ \delta(x_1-x_2),\\
C(\gamma_2)&=-\langle\Delta^2(z-x_4)\,,\,\mathrm{w}_{\gamma_2}(z-x_4)\rangle_z\ \delta(x_3-x_4),
\end{aligned}\tag{68}$$

and according to the choice $\mathrm{w}_{\gamma_1\uplus\gamma_2}=\mathrm{w}_{\gamma_1}\mathrm{w}_{\gamma_2}$

$$C(\gamma_1\uplus\gamma_2)=\langle\Delta^2(y-x_2)\,,\,\mathrm{w}_{\gamma_1}(y-x_2)\rangle_y\langle\Delta^2(z-x_4)\,,\,\mathrm{w}_{\gamma_2}(z-x_4)\rangle_z\delta(x_1-x_2)\delta(x_3-x_4), \tag{69}$$

and thus

$$C(\gamma_1\uplus\gamma_2)=C(\gamma_1)C(\gamma_2). \tag{70}$$

Note that this is *not* a definition: it is the result of a calculation in which the recursive definition has been exclusively used.

The previous example was simple in the sense that all divergences encountered were logarithmic. An example with quadratic (sub)divergences is given by the diagram (c) in Figure 7. One generalizes the factorization results of the examples in the following lemma.

Lemma 5.1 *The counterterm map C verifies $C(\gamma_1 \uplus \cdots \uplus \gamma_n) = C(\gamma_1) \ldots C(\gamma_n)$.*

Proof. We argue by induction; the previous example and example (c) in Section 7 show how to start it. We have:

$$(-S_\gamma \bar{R}_\gamma)\, f(\gamma) = -S_\gamma \Bigg(\sum_{\emptyset \subseteq \tilde{\gamma} \subsetneq \gamma} [-S_{\tilde{\gamma}_1} \bar{R}_{\tilde{\gamma}_1}] \cdots [-S_{\tilde{\gamma}_n} \bar{R}_{\tilde{\gamma}_n}] \Bigg) f(\gamma). \tag{71}$$

In order to obtain and interpret this formula, proceed as follows: use the definition of $\bar{R}_\gamma$ in terms of the subgraphs $\tilde{\gamma} \subset \gamma$. Now, by the induction hypothesis, the lemma is true for the $\tilde{\gamma}$. We apportion $\tilde{\gamma}$ according to the pieces of γ. A connected piece of a given $\tilde{\gamma}$ can belong to one of the $\gamma_1 \cdots \gamma_n$ only; but there could be several connected pieces of $\tilde{\gamma}$ in any of the $\gamma_1 \cdots \gamma_n$, say in γ_i. Using the induction hypothesis (read in reverse) we treat them as a unity; call it $\tilde{\gamma}_i$. Finally, if there is no connected piece of $\tilde{\gamma}$ in, say, γ_k, for $k \in \{1 \cdots n\}$, we pretend that there is the piece $\emptyset = \tilde{\gamma}_k \subset \gamma_k$.

The previous formula then we rewrite as

$$-S_\gamma \Bigg(\Bigg[\sum_{\emptyset \subseteq \tilde{\gamma}_1 \subseteq \gamma_1} (-S_{\tilde{\gamma}_1} \bar{R}_{\tilde{\gamma}_1}) \Bigg] \cdots \Bigg[\sum_{\emptyset \subseteq \tilde{\gamma}_n \subseteq \gamma_n} (-S_{\tilde{\gamma}_n} \bar{R}_{\tilde{\gamma}_n}) \Bigg] - (-S_{\gamma_1} \bar{R}_{\gamma_1}) \cdots (-S_{\gamma_n} \bar{R}_{\gamma_n}) \Bigg) f(\gamma), \tag{72}$$

by adding and subtracting the last term. Now, by the same definition:

$$\sum_{\emptyset \subseteq \tilde{\gamma}_i \subseteq \gamma_i} (-S_{\tilde{\gamma}_i} \bar{R}_{\tilde{\gamma}_i}) f(\gamma_i) = (1 - S_{\gamma_i}) \bar{R}_{\gamma_i} f(\gamma_i). \tag{73}$$

Therefore, the expression is further transformed into

$$-S_\gamma \Bigg([(1 - S_{\gamma_1}) \bar{R}_{\gamma_1}] \cdots [(1 - S_{\gamma_n}) \bar{R}_{\gamma_n}] - (-S_{\gamma_1} \bar{R}_{\gamma_1}) \ldots (-S_{\gamma_n} \bar{R}_{\gamma_n}) \Bigg) f(\gamma). \tag{74}$$

Because the degree of divergence of γ is the sum of the degree of divergence of its connected parts, the first term vanishes under the factorization rule for the auxiliary functions (a slight generalization of Lemma 4.3). Furthermore, $S_\gamma S_{\gamma_1} \cdots S_{\gamma_n} = S_{\gamma_1} \cdots S_{\gamma_n}$. $\square$

The combinatorics of this proof is identical to the one in the parallel proof for the BPHZ formalism [41]. It is well known, and fairly clear from the examples in this section and Section 7, that the recursive formula for renormalization, for a proper connected diagram Γ, gives rise to the following nonrecursive formula:

$$R_\Gamma f(\Gamma) = \Big[1 + \sum_{\mathcal{E}} \prod_{\gamma \in \mathcal{E}} (-S_\gamma) \Big] f(\Gamma), \tag{75}$$

where one sums over all nonempty sets $\mathcal{E}$ whose elements γ are proper, divergent, not necessarily connected subdiagrams made of subgraphs of Γ, that may be Γ itself, and $\gamma_1, \gamma_2 \in \mathcal{E}$ implies that $\gamma_1 \subsetneq \gamma_2$ or vice versa. If $\gamma_1 \subsetneq \gamma_2$, then the order of the subtractions is as $\cdots S_{\gamma_2} \cdots S_{\gamma_1} \cdots$. It is also rather immediate here, as a consequence of the lemma, that the previous formula can be rewritten as

$$R_\Gamma f(\Gamma) = \Big[1 + \sum_{\mathcal{F}} \prod_{\gamma \in \mathcal{F}} (-S_\gamma) \Big] f(\Gamma), \tag{76}$$

with $\mathcal{F}$ now denoting nonempty sets whose elements γ are proper, divergent, and *connected* subgraphs of Γ, that may be Γ itself, and $\gamma_1, \gamma_2 \in \mathcal{F}$ implies that either $\gamma_1 \subsetneq \gamma_2$ or $\gamma_2 \subsetneq \gamma_1$ *or* that γ_1, γ_2 are disjoint; the order of subtractions is as before.

In other words, we now sum over forests in the sense of Zimmermann, the difference with [42] being that, since we are in real space, we sum over subgraphs instead of over subdiagrams. That the forest formula in this sense holds in the context of the Epstein–Glaser renormalization must be known to the experts, but we were unable to find an argument for that in the literature.

For theoretical purposes, there is some advantage in the forest formula, e.g., for the notorious QED diagram mooted by Kreimer in [2] there are 32 $\mathcal{F}$-type sets, whereas there are 68 $\mathcal{E}$-type sets. For practical purposes, one of course always uses the recursive formulae à la Bogoliubov–Epstein–Glaser.

(The more trivial cases of renormalization are easily handled. When Γ with $E > 4$ is not primitive, we set

$$R_\Gamma f(\Gamma) = \bar{R}_\Gamma f(\Gamma). \tag{77}$$

If Γ is primitive, then simply

$$R_\Gamma f(\Gamma) = W_\Gamma f(\Gamma), \tag{78}$$

i.e., $\bar{R}_\Gamma = 1$. Finally, we can let in fully convergent diagrams by decreeing that, if $f(\Gamma)$ is superficially convergent and has no subdivergence, simply

$$R_\Gamma f(\Gamma) = f(\Gamma).) \tag{79}$$

6 A Hopf algebra of graphs for the ϕ_4^4 theory

Now we are all set to prove the compatibility of CPT with the essential kernel of the Hopf algebra picture of Connes and Kreimer. We start with a preliminary result. If $\gamma' \subseteq \gamma \subseteq \Gamma$, then γ/γ' is naturally interpreted as a subdiagram of Γ/γ' and there is the following obvious

Lemma 6.1 *If $\gamma' \subseteq \gamma \subseteq \Gamma$, then $(\Gamma/\gamma')/(\gamma/\gamma') \simeq \Gamma/\gamma$.*

A point not emphasized in [5] is that the definition of the Hopf algebra of Feynman graphs associated to a given quantum field theory can be made in slightly different ways, depending on purpose. In distinction to reference [5], we allow for improper diagrams living in our Hopf algebra. Namely, connected graphs will be considered as the generators instead of only proper ones as in [5]. (In fact, it is feasible to allow for *all* Feynman graphs appearing in the expansion of $\mathbf{S}$.) Consideration of improper diagrams is called for in the Epstein–Glaser formalism, that leads naturally to connected Green functions and not to 1-particle irreducible ones.

That said, the algebra $\mathcal{H}$ is defined as the polynomial (hence commutative) algebra generated by the empty set $\emptyset$ and the connected Feynman graphs that are (superficially) divergent and/or have (superficially) divergent subdiagrams, with set union as the product operation (hence $\emptyset$ is the unit element of $\mathcal{H}$).

A telling operation in a Hopf algebra $\mathcal{H}$ is often the coproduct $\triangle : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$; as it is to be a homomorphism of the algebra structure, we need only define it on connected diagrams. By definition, the coproduct of Γ, a graph in ϕ_4^4 theory, is given by

$$\triangle\Gamma = \sum_{\emptyset \subseteq \gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma \,. \tag{80}$$

The sum is over all divergent, proper, not necessarily connected subdiagrams of Γ that are subgraphs or products of subgraphs, such that *each piece is divergent*, including (and then with the possible exception of, as Γ need not be divergent nor proper) the empty set and Γ itself —as the formula makes explicit. If $\gamma = \emptyset$, we write $\gamma = 1$ and $\Gamma/\gamma = \Gamma$; if $\gamma = \Gamma$, we write $\Gamma/\gamma = 1$. Note that a nonempty Γ/γ will be proper iff Γ is —the situation considered in [5].

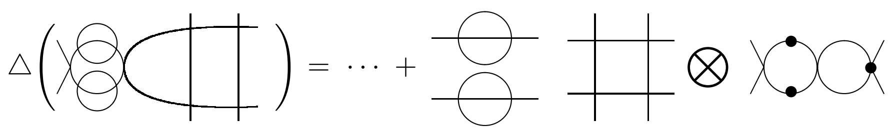


Figure 5: A vanishing contribution of a logarithmically divergent disconnected subdiagram.

Note that the definition of the Hopf algebra coproduct excludes terms like the one indicated in Figure 5, that would give anyway zero contribution because of Lemma 5.1. This also guarantees that the graphs Γ and Γ/γ have the same external structure.

If there are no nontrivial subdiagrams in Γ, then this graph will be primitive both in the sense of quantum field and Hopf algebra theory.

In Figure 6 there are some examples.

The counit $\varepsilon : \mathcal{H} \to \mathbb{C}$ is given by $\varepsilon(\Gamma) = 0$ for all graphs and $\varepsilon(\emptyset) = 1$. All of the bialgebra axioms are readily verified, except for coassociativity. The coproduct in our Hopf algebra of Feynman graphs governs in some sense the various splittings of a T-product at a given order in terms of lower order T-products (see the criterion towards the end of Section 4), and so coassociativity is of course the soul of the whole matter. But this is easy enough.

Lemma 6.2 *The algebra of graphs $\mathcal{H}$ is a Hopf algebra.*

Proof. We bother only with the proof of coassociativity. We need to show that

$$((\triangle \otimes \mathrm{id}) \circ \triangle)\Gamma = ((\mathrm{id} \otimes \triangle) \circ \triangle)\Gamma,$$

for every connected graph Γ, where id denotes the identity map of $\mathcal{H}$ into itself. Using the definition, the left hand side is written as

$$\sum_{\emptyset \subseteq \gamma' \subseteq \gamma \subseteq \Gamma} \gamma' \otimes \gamma/\gamma' \otimes \Gamma/\gamma,$$

and the right hand side as

$$\sum_{\emptyset \subseteq \gamma' \subseteq \Gamma,\, \emptyset \subseteq \gamma'' \subseteq \Gamma/\gamma'} \gamma' \otimes \gamma'' \otimes (\Gamma/\gamma')/\gamma''.$$

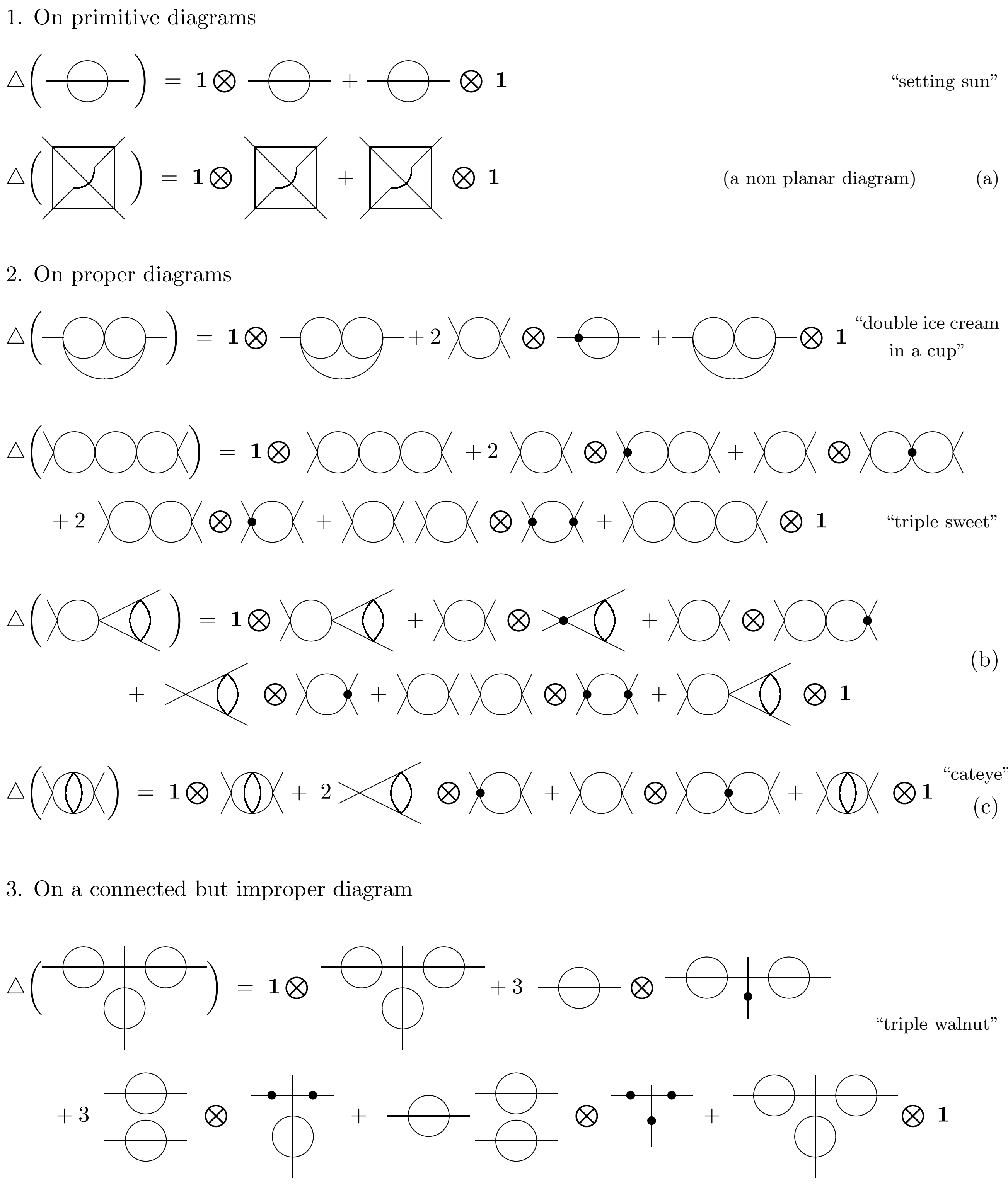


Figure 6: A few examples of coproducts.

We must then show that

$$\sum_{\gamma'\subseteq\gamma\subseteq\Gamma} \gamma/\gamma' \otimes \Gamma/\gamma = \sum_{\emptyset\subseteq\gamma''\subseteq\Gamma/\gamma'} \gamma'' \otimes (\Gamma/\gamma')/\gamma''.$$

Let $\gamma' \subseteq \gamma \subseteq \Gamma$. Then, as already remarked, $\emptyset \subseteq \gamma/\gamma' \subseteq \Gamma/\gamma'$. Reciprocally, to every $\gamma'' \subseteq \Gamma/\gamma'$ corresponds a γ such that $\gamma' \subseteq \gamma \subseteq \Gamma$ and $\gamma/\gamma' = \gamma''$. Then use Lemma 6.1. □

Note that if $\#(\Gamma)$ denotes the number of vertices in Γ, then $\mathcal{H}$ is graded by $\# - 1$. To be precise, the degree of a generator Γ (connected element) is $\#(\Gamma) - 1$; the degree of a product is the sum of the degrees of the factors. This grading is compatible with the coproduct. For graded bialgebras, it is easy to construct the antipode, by methods spelled out in [23, Ch. 14], for instance; and so $\mathcal{H}$ is a Hopf algebra.

The main role for the definition of coproduct in a coalgebra is that it allows to define a convolution operation $\star$ for two linear maps f, g to any algebra, given by the composition

$$g \star f\ (h) = \sum g(h')f(h'') \tag{81}$$

if $\triangle h = \sum h' \otimes h''$. In particular, for our (commutative) Hopf algebra definition (80),

$$g \star f\ (\Gamma) = g(\Gamma) + f(\Gamma) + \sum_{\emptyset\subsetneq\gamma\subsetneq\Gamma} g(\gamma)\, f(\Gamma/\gamma), \tag{82}$$

and the convolution of two homomorphisms will be a homomorphism. Now, because of lemma 5.1 in the previous section, the linear map C is multiplicative. Thus the renormalization formula (54)

$$R_\Gamma f(\Gamma) =: R(\Gamma) = C(\Gamma) + f(\Gamma) + \sum_{\emptyset\subsetneq\gamma\subsetneq\Gamma} C(\gamma)\, f(\Gamma/\gamma) \tag{83}$$

is recast as

$$R = C \star f, \tag{84}$$

dropping the variable tag Γ, as we now may. In summary, the main result of this paper can be stated as follows.

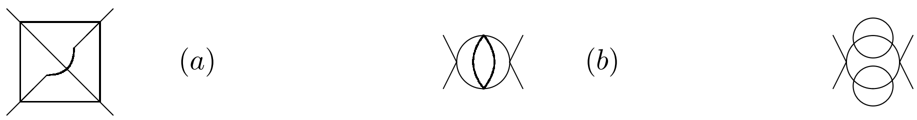
Theorem 6.1 *Bogoliubov's renormalization maps R are (distributional) characters on the Hopf algebra of graphs $\mathcal{H}$. They are the Hopf convolution of the (unrenormalized) Feynman graph character and a counterterm character.*

This is what opens the door to the application of Lie-theoretical methods characteristic of (co)-commutative Hopf algebra theory in the renormalized theory [7, 9]. In particular, it is clear that R-maps differing in the choice of subtraction operators are related by elements of a huge "renormalization group" sitting inside $\mathcal{G}_\mathcal{H}$. In other words, having constructed two solutions for the scattering operator involving two different T-products, the relationship between their respective "coupling constants" is given by a local formal diffeomorphism [29].

7 Some further examples of CPT renormalization

In this Section, we shall renormalize a few other diagrams in ϕ^4_4 theory along the lines prescribed by the Epstein–Glaser scheme according to the previous sections.

We shall be concerned with the three following graphs, see Figure 7, two of order 4 ((a) and (c) of Figure 6) and one of order 6.



(a) $:\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4):$

(b) $:\phi^2(x_1)\phi^2(x_4):$

(c) $:\phi^2(x_1)\phi^2(x_4):$

Figure 7: An interesting primitive graph (a), overlapping logarithmic subdivergences (b) and quadratic subdivergences, the "cafard" (c).

All are logarithmically superficially divergent, $\omega(\Gamma) = 4 - E = 0$, although the last one has quadratic subdivergences. They come in $\mathbf{S}$ as coefficients of the Wick monomials $:\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4):$ for (a) and $:\phi^2(x_1)\phi^2(x_4):$ for (b) and (c). We shall not care about the symmetry factors. In the computation the criterion (51) for the Epstein–Glaser recursive construction will be applied but only for partitionings associated to divergent, proper, not necessarily connected subgraphs.

Graph (a)

Numbering the vertices clockwise, one gets the operator

$$\langle R_\Gamma f(\Gamma)\,,\,\varphi_\Gamma\rangle$$
$$\propto\ \lambda^4\ \langle[\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta(x_1-x_4)\Delta(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_2)]_R\,,\,\varphi_\Gamma(x_1,x_2,x_3,x_4)\rangle,$$

where $\varphi_\Gamma(x_1,x_2,x_3,x_4)$ stands for $:\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4):\ g(x_1)g(x_2)g(x_3)g(x_4)$. According to (78) one obtains

$$\begin{aligned}\langle\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta(x_1-x_4)\Delta(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_2),&\\ :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4):\ g(x_1)g(x_2)g(x_3)g(x_4)&\\ -\ w_\Gamma(x_1-x_2,x_2-x_3,x_3-x_4):\phi^4(x_4):g^4(x_4)\rangle&\end{aligned}$$

corresponding to the following pattern of subtraction:

$$f(\Gamma)\mapsto(1-S_\Gamma)\,f(\Gamma). \tag{85}$$

Graph (b)

Numbering the vertices from left to right, this corresponds to the operator

$$\langle R_\Gamma f(\Gamma)\,,\,\varphi_\Gamma\rangle$$

$$\propto\ \lambda^4\ \langle[\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta^2(x_2-x_3)\Delta(x_2-x_4)\Delta(x_3-x_4)]_R\,,\ \varphi_\Gamma(x_1,x_2,x_3,x_4)\rangle,$$

where $\varphi_\Gamma(x_1,x_2,x_3,x_4)$ stands for $:\!\phi^2(x_1)\phi^2(x_4)\!:\ g(x_1)g(x_2)g(x_3)g(x_4)$. Therefore,

$$\begin{aligned}\langle\bar R_\Gamma f(\Gamma)\,,\,W_\Gamma\varphi_\Gamma\rangle &= \langle[\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta^2(x_2-x_3)\Delta(x_2-x_4)\Delta(x_3-x_4)]_{\bar R}\,,\\ &:\!\phi^2(x_1)\phi^2(x_4)\!:\ g(x_1)g(x_2)g(x_3)g(x_4)-w_\Gamma(x_1-x_2,x_2-x_3,x_3-x_4):\!\phi^4(x_4)\!:g^4(x_4)\rangle.\end{aligned}$$

Taking into account all three logarithmically divergent subgraphs, the whole ice-cream cones on the left γ_1 and on the right γ_2, and the central fish $\gamma_3=\gamma_1\cap\gamma_2$, which is the overlap of the former,

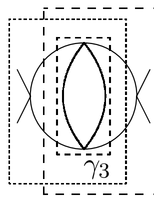


Figure 8: Identification of proper subdiagrams with an overlapping logarithmic divergence.

one successively gets

$$\begin{aligned}&\langle\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta^2(x_2-x_3)\Delta(x_2-x_4)\Delta(x_3-x_4),:\!\phi^2(x_1)\phi^2(x_4)\!:\ g(x_1)g(x_2)g(x_3)g(x_4)\\ &\qquad\qquad -\ w_\Gamma(x_1-x_2,x_2-x_3,x_3-x_4):\!\phi^4(x_4)\!:g^4(x_4)\rangle\\ &-\ \langle[\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta^2(x_2-x_3)]_{\bar R}\Delta^2(x_3-x_4),\\ &\qquad\qquad w_{\gamma_1}(x_1-x_2,x_2-x_3):\!\phi^2(x_1)\phi^2(x_4)\!:\ g^3(x_1)g(x_4)\\ &\qquad\qquad -w_{\gamma_1}(x_1-x_2,x_2-x_3)w_\Gamma(0,0,x_3-x_4):\!\phi^4(x_4)\!:g^4(x_4)\rangle \qquad (86)\\ &-\ \langle[\Delta^2(x_2-x_3)\Delta(x_2-x_4)\Delta(x_3-x_4)]_{\bar R}\Delta^2(x_1-x_2),\\ &\qquad\qquad w_{\gamma_2}(x_2-x_3,x_3-x_4):\!\phi^2(x_1)\phi^2(x_4)\!:\ g(x_1)g^3(x_4)\\ &\qquad\qquad -w_{\gamma_2}(x_2-x_3,x_3-x_4)w_\Gamma(x_1-x_2,0,0):\!\phi^4(x_4)\!:g^4(x_4)\rangle \qquad (87)\\ &-\ \langle\Delta^2(x_1-x_2)\Delta^2(x_2-x_3)\Delta^2(x_3-x_4),w_{\gamma_3}(x_2-x_3):\!\phi^2(x_1)\phi^2(x_4)\!:\ g(x_1)g^2(x_2)g(x_4)\\ &\qquad\qquad -\ w_{\gamma_3}(x_2-x_3)w_\Gamma(x_1-x_2,0,x_3-x_4):\!\phi^4(x_4)\!:g^4(x_4)\rangle.\end{aligned}$$

Incorporating the renormalization of the subgraphs γ_1 and γ_2 (previously obtained at the lower order 3) yields the replacement of the pairings (86) and (87) respectively by

$$\begin{aligned}(86) = &-\langle\Delta(x_1-x_2)\Delta(x_1-x_3)\Delta^2(x_2-x_3)\Delta^2(x_3-x_4),\\ &\qquad w_{\gamma_1}(x_1-x_2,x_2-x_3):\!\phi^2(x_1)\phi^2(x_4)\!:\ g^3(x_1)g(x_4)\\ &\qquad -w_{\gamma_1}(x_1-x_2,x_2-x_3)w_\Gamma(0,0,x_3-x_4):\!\phi^4(x_4)\!:g^4(x_4)\rangle\\ &+\langle\Delta^2(x_1-x_2)\Delta^2(x_2-x_3)\Delta^2(x_3-x_4),w_{\gamma_3}(x_2-x_3)w_{\gamma_1}(x_1-x_2,0):\!\phi^2(x_1)\phi^2(x_4)\!:\ g^3(x_1)g(x_4)\\ &\qquad -w_{\gamma_3}(x_2-x_3)w_{\gamma_1}(x_1-x_2,x_2-x_3)w_\Gamma(0,0,x_3-x_4):\!\phi^4(x_4)\!:g^4(x_4)\rangle\end{aligned}$$

$$\begin{aligned}(87) = &-\langle\Delta^2(x_1-x_2)\Delta^2(x_2-x_3)\Delta(x_2-x_4)\Delta(x_3-x_4),\\ &\qquad w_{\gamma_2}(x_2-x_3,x_3-x_4):\!\phi^2(x_1)\phi^2(x_4)\!:\ g(x_1)g^3(x_4)\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{w}_{\gamma_2}(x_2-x_3,x_3-x_4)\mathrm{w}_\Gamma(x_1-x_2,0,0){:}\phi^4(x_4){:}g^4(x_4)\rangle \\
&+\langle\Delta^2(x_1-x_2)\Delta^2(x_2-x_3)\Delta^2(x_3-x_4),\, \mathrm{w}_{\gamma_3}(x_2-x_3)\mathrm{w}_{\gamma_2}(0,x_3-x_4){:}\phi^2(x_1)\phi^2(x_4){:}\; g(x_1)g^3(x_4) \\
&-\mathrm{w}_{\gamma_3}(x_2-x_3)\mathrm{w}_{\gamma_2}(0,x_3-x_4)\mathrm{w}_\Gamma(x_1-x_2,0,0){:}\phi^4(x_4){:}g^4(x_4)\rangle.
\end{aligned}
$$

This graph has the following pattern of subtraction:

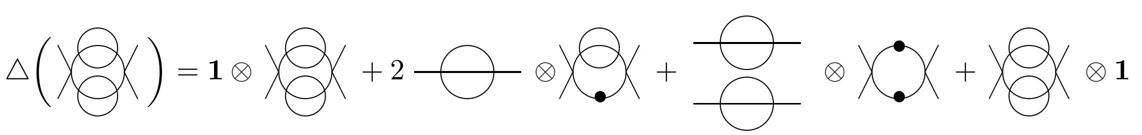
$$
f(\Gamma)\mapsto(1-S_\Gamma)[1-S_{\gamma_1}(1-S_{\gamma_3})-S_{\gamma_2}(1-S_{\gamma_3})-S_{\gamma_3}]\,f(\Gamma)\;. \tag{88}
$$

Graph (c)

Among all the operators contributing to the sixth order one considers the one associated to the "cafard" graph. It is mathematically convenient, for this kind of diagrams, to assume that all auxiliary functions used have vanishing derivatives at the coincidence points up to the highest order of divergence encountered in the diagram (failure to do so can always be corrected by an appropriate choice of finite counterterms). Also as in [21], it is recalled that when some "coupling constants" g become derivated according to the W projection, the corresponding terms vanish in the adiabatic limit (in the strong sense in massive ϕ^4_4 theory). Thus such terms can be discarded. Numbering the six vertices clockwise, one has the operator

$$
\begin{aligned}
\langle R_\Gamma f(\Gamma)\,,\,\varphi_\Gamma\rangle \;\propto\; \lambda^6\;\langle[\Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_5)\Delta^3(x_5-x_6)\Delta(x_6-x_1)]_R, \\
\varphi_\Gamma(x_1,x_2,x_3,x_4,x_5,x_6)\rangle,
\end{aligned} \tag{89}
$$

where $\varphi_\Gamma(x_1,x_2,x_3,x_4,x_5,x_6)$ stands for ${:}\phi^2(x_1)\phi^2(x_4){:}g(x_1)g(x_2)g(x_3)g(x_4)g(x_5)g(x_6)$. There are three graphs to be considered, namely, both the top and bottom sunsets γ_1 and γ_2 respectively which are quadratically divergent, $\omega(\gamma_i)=2$, $i=1,2$ and the union of the two $\gamma_1\uplus\gamma_2$ with power counting index $\omega(\gamma_1\uplus\gamma_2)=\omega(\gamma_1)+\omega(\gamma_2)=4$ according to the coproduct:



So (89) reads

$$
\begin{aligned}
\langle\bar R_\Gamma f(\Gamma)\,,\,W_\Gamma\varphi_\Gamma\rangle &= \langle[\Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_5)\Delta^3(x_5-x_6)\Delta(x_6-x_1)]_{\bar R}\,, \\
&\quad {:}\phi^2(x_1)\phi^2(x_4){:}\; g(x_1)g(x_2)g(x_3)g(x_4)g(x_5)g(x_6) \\
&\quad -\mathrm{w}_\Gamma(x_1-x_2,x_2-x_3,x_3-x_4,x_4-x_5,x_5-x_6){:}\phi^4(x_4){:}g^6(x_4)\rangle.
\end{aligned}
$$

Then we perform the renormalization by the use of previously renormalized graphs according to their respective W projection. For notational convenience, the pairing will be done with the projection $W_\Gamma\varphi_\Gamma$ and $\Delta^{(a)}$ below will denote the derivative of order $|a|$ of the propagator with respect to the multi-quadri-index a, so the previous expression becomes

$$
\langle\Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_5)\Delta^3(x_5-x_6)\Delta(x_6-x_1)\,,W_\Gamma\varphi_\Gamma\rangle
$$

$$
\begin{aligned}
&- \sum_{|a|=0}^{2} \sum_{\substack{|b|,|c|=0 \\ b+c=a}}^{|a|} \Big\langle \Delta^{(b)}(x_3-x_1)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_5)\Delta^3(x_5-x_6)\Delta(x_1-x_6)\,, \\
&\qquad\qquad w_{\gamma_1}(x_2-x_3)\,\frac{(x_2-x_3)^a}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^c}\right)\Big|_{x_2=x_3}\Big\rangle \\
&- \sum_{|a|=0}^{2} \sum_{\substack{|b|,|c|=0 \\ b+c=a}}^{|a|} \Big\langle \Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta^{(b)}(x_6-x_4)\Delta^3(x_5-x_6)\Delta(x_1-x_6)\,, \\
&\qquad\qquad w_{\gamma_2}(x_5-x_6)\,\frac{(x_5-x_6)^a}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_5^c}\right)\Big|_{x_5=x_6}\Big\rangle \\
&- \sum_{\substack{|a|=0 \\ a=a_1+a_2 \\ |a_1|,|a_2|\le 2}}^{4} \sum_{\substack{|b|=|b_1|+|b_2|=0 \\ |c|=|c_1|+|c_2|=0 \\ b+c=a}}^{|a|} \Big\langle [\Delta^3(x_2-x_3)\Delta^3(x_5-x_6)]_{\bar R} \\
&\qquad\qquad \Delta^{(b_1)}(x_3-x_1)\Delta(x_3-x_4)\Delta^{(b_2)}(x_6-x_4)\Delta(x_6-x_1)\,, \\
&w_{\gamma_1\uplus\gamma_2}(x_2-x_3,x_5-x_6)\,\frac{(x_2-x_3)^{a_1}(x_5-x_6)^{a_2}}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^{c_1}\partial x_5^{c_2}}\right)\Big|_{\substack{x_2=x_3 \\ x_5=x_6}}\Big\rangle
\end{aligned} \tag{90}
$$

In the last pairing (90), the summation on the multi-quadri-index $a=(a_1,a_2)$ is restricted to the condition $|a_1|,|a_2|\le 2$ due to the fact that $\gamma_1\uplus\gamma_2$ is a disconnected graph. For instance, pairing the tensor product distribution $[\Delta^3(x_2-x_3)\Delta^3(x_5-x_6)]_{\bar R}$ with a test function $\varphi(x_2-x_3)$ yields a distribution in x_5-x_6 of singular order 2. Then finally (90) is worked out by using the renormalized quadratically divergent subgraphs, as follows:

$$
\begin{aligned}
(90) = &- \sum_{\substack{|a|=0 \\ a=a_1+a_2 \\ |a_1|,|a_2|\le 2}}^{4} \sum_{\substack{|b|=|b_1|+|b_2|=0 \\ |c|=|c_1|+|c_2|=0 \\ b+c=a}}^{|a|} \Big\langle \Delta^3(x_2-x_3)\Delta^3(x_5-x_6) \\
&\qquad\qquad \Delta^{(b_1)}(x_3-x_1)\Delta(x_3-x_4)\Delta^{(b_2)}(x_6-x_4)\Delta(x_6-x_1)\,, \\
&w_{\gamma_1\uplus\gamma_2}(x_2-x_3,x_5-x_6)\,\frac{(x_2-x_3)^{a_1}(x_5-x_6)^{a_2}}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^{c_1}\partial x_5^{c_2}}\right)\Big|_{\substack{x_2=x_3 \\ x_5=x_6}} \\
&-w_{\gamma_1}(x_2-x_3)\sum_{|a'|=0}^{2}\frac{(x_2-x_3)^{a'}(x_5-x_6)^{a_2}}{b!\,c!\,a'!}\underbrace{\left[\frac{\partial^{|a'|}[w_{\gamma_1\uplus\gamma_2}(x_2-x_3,x_5-x_6)(x_2-x_3)^{a_1}]}{\partial x_2^{a'}}\right]\Big|_{x_2=x_3}}_{a_1!\,w_{\gamma_1\uplus\gamma_2}(0,x_5-x_6)\,\delta^{a_1}_{a'}} \\
&\qquad\qquad \left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^{c_1}\partial x_5^{c_2}}\right)\Big|_{\substack{x_2=x_3 \\ x_5=x_6}} \\
&-w_{\gamma_2}(x_5-x_6)\sum_{|a'|=0}^{2}\frac{(x_5-x_6)^{a'}(x_2-x_3)^{a_1}}{b!\,c!\,a'!}\underbrace{\left[\frac{\partial^{|a'|}[w_{\gamma_1\uplus\gamma_2}(x_2-x_3,x_5-x_6)(x_5-x_6)^{a_2}]}{\partial x_5^{a'}}\right]\Big|_{x_5=x_6}}_{a_2!\,w_{\gamma_1\uplus\gamma_2}(x_2-x_3,0)\,\delta^{a_2}_{a'}}
\end{aligned}
$$

$$\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^{c_1}\partial x_5^{c_2}}\right)\Big|_{\substack{x_2=x_3\\x_5=x_6}}\Big\rangle.$$

Remembering the judicious choice $w_{\gamma_1\uplus\gamma_2}=w_{\gamma_1}w_{\gamma_2}$, the first term is compensated by one of the two last terms, so that for (89) one ends with

$$\begin{aligned}
&\langle\Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_5)\Delta^3(x_5-x_6)\Delta(x_6-x_1)\,,W_\Gamma\varphi_\Gamma\rangle\\
&\quad-\sum_{|a|=0}^{2}\sum_{\substack{|b|,|c|=0\\b+c=a}}^{|a|}\Big\langle\Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta(x_4-x_5)\Delta^3(x_5-x_6)\Delta(x_1-x_6)\,,\\
&\qquad\qquad w_{\gamma_1}(x_2-x_3)\,\frac{(x_2-x_3)^a}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^c}\right)\Big|_{x_2=x_3}\Big\rangle\\
&\quad-\sum_{|a|=0}^{2}\sum_{\substack{|b|,|c|=0\\b+c=a}}^{|a|}\Big\langle\Delta(x_1-x_2)\Delta^3(x_2-x_3)\Delta(x_3-x_4)\Delta^{(b)}(x_6-x_4)\Delta^3(x_5-x_6)\Delta(x_1-x_6)\,,\\
&\qquad\qquad w_{\gamma_2}(x_5-x_6)\,\frac{(x_5-x_6)^a}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_5^c}\right)\Big|_{x_5=x_6}\Big\rangle\\
&\quad+\sum_{\substack{|a|=0\\a=a_1+a_2\\|a_1|,|a_2|\le 2}}^{4}\ \sum_{\substack{|b|=|b_1|+|b_2|=0\\|c|=|c_1|+|c_2|=0\\b+c=a}}^{|a|}\Big\langle\Delta^3(x_2-x_3)\Delta^3(x_5-x_6)\\
&\qquad\qquad\Delta^{(b_1)}(x_3-x_1)\Delta(x_3-x_4)\Delta^{(b_2)}(x_6-x_4)\Delta(x_6-x_1)\,,\\
&\quad w_{\gamma_1}(x_2-x_3)w_{\gamma_2}(x_5-x_6)\,\frac{(x_2-x_3)^{a_1}(x_5-x_6)^{a_2}}{b!\,c!}\left(\frac{\partial^{|c|}W_\Gamma\varphi_\Gamma}{\partial x_2^{c_1}\partial x_5^{c_2}}\right)\Big|_{\substack{x_2=x_3\\x_5=x_6}}\Big\rangle\,,
\end{aligned}$$

a pairing which gives rise to the following subtraction pattern:

$$f(\Gamma)\mapsto(1-S_\Gamma)[1-S_{\gamma_1}-S_{\gamma_2}+S_{\gamma_1}S_{\gamma_2}]\,f(\Gamma)\ . \tag{91}$$

Accordingly, counterterms all come from connected diagrams, and we see immediately that

$$C(\gamma_1)=-\sum_{|a_1|=0}^{2}\Big\langle\Delta^3(y-x_3),\,w_{\gamma_1}(y-x_3)\frac{(y-x_3)^{a_1}}{a_1!}\Big\rangle_y(-1)^{|a_1|}\delta^{(a_1)}(x_2-x_3), \tag{92}$$

and

$$C(\gamma_2)=-\sum_{|a_2|=0}^{2}\Big\langle\Delta^3(z-x_6),\,w_{\gamma_2}(z-x_6)\frac{(z-x_6)^{a_2}}{a_2!}\Big\rangle_z(-1)^{|a_2|}\delta^{(a_2)}(x_5-x_6). \tag{93}$$

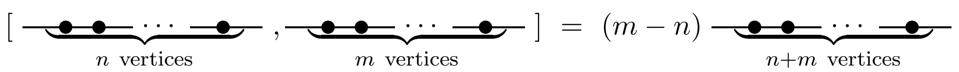
It is readily seen that $C(\gamma_1\uplus\gamma_2)=C(\gamma_1)C(\gamma_2)$ since

$$\begin{aligned}
C(\gamma_1\uplus\gamma_2)=\sum_{\substack{|a|=0\\a=a_1+a_2\\|a_1|,|a_2|\le 2}}^{4}&\Big\langle\Delta^3(y-x_3),\,w_{\gamma_1}(y-x_3)\frac{(y-x_3)^{a_1}}{a_1!}\Big\rangle_y\\
&\Big\langle\Delta^3(z-x_6),\,w_{\gamma_2}(z-x_6)\frac{(z-x_6)^{a_2}}{a_2!}\Big\rangle_z(-1)^{|a|}\delta^{(a_1)}(x_2-x_3)\,\delta^{(a_2)}(x_5-x_6).
\end{aligned}$$

8 Conclusions

Despite its elegance and accuracy, the CPT scheme of Epstein and Glaser was thought not to yield explicit formulae of actual computational value. Nevertheless, the Zürich School kept the flame; we want to cite here, apart from the textbook by Scharf [17] and [34], the outstanding paper by Hurth and Skenderis [43] on CPT for gauge theories, among its more recent productions. Also the Moscow INR School for a long while made systematic use of distributional methods; in this respect [44], for instance, makes fascinating reading. The computational links highlighted by people from the Hamburg School [16, 18, 21] have contributed, to change the picture. It is important to realize that CPT suffers none of the limitations of dimensional regularization (for a nonstandard example, see [45]) and that it has naturally associated to it regularization schemes that comprise the MS scheme [46]. On the other hand, the Hopf algebra approach points to a continent of symmetry lying beyond the sea mists of the present formalism of quantum field theory [9]; the versatility of CPT is surely an important asset for unveiling that continent of symmetry.

The change of renormalization scheme leads us to the use of subgraphs, rather than more general subdiagrams; other differences between our method of proof and the treatment in [5] by Connes and Kreimer are wholly superficial: for instance, the fact that they had a particular regularization at their disposal, allowed them to formulate the basic argument in a more algebraic way, whereas in our case it boils down to concrete analytical properties of the maps W of Epstein and Glaser. Another minor difference concerns the place accorded here to improper diagrams. Let us note, nevertheless, that a congeries of improper diagrams realize the Connes–Moscovici Lie algebra [8] relation $[Z_m, Z_n] = (m-n)Z_{m+n}$ inside our Hopf algebra $\mathcal{H}$ [47]. This commutator can be graphically represented as



(94)

"Real life" theories, i.e., gauge theories, also represent a challenge for the Hopf algebra approach to the renormalization group. In real life, some of our arguments may not strictly apply; for instance, normalization conditions or the convenience of preserving gauge invariance at some stage of the procedure might demand "oversubtractions", which effectively represent an increase of the order of divergence of some diagrams, propagating itself to higher-order diagrams. It looks likely, however, that these technical difficulties will be overcome.

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